

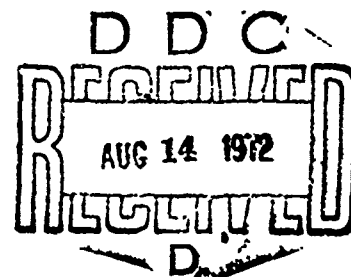
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(11)
NOLTR 72-80

CALCULATION OF MAGNUS FORCES ON
AXISYMMETRIC BODIES AT SMALL ANGLES
OF ATTACK WITH INCOMPRESSIBLE
TURBULENT BOUNDARY LAYERS

By
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21 MARCH 1972



NOL

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TECHNICAL
REPORT

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DISTRIBUTION UNLIMITED

NOLTR 72-80

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Ordnance Laboratory Silver Spring, Maryland 20910		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE Calculation of Magnus Forces on Axisymmetric Bodies at Small Angles of Attack with Incompressible Turbulent Boundary Layers			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)			
5. AUTHOR(S) (First name, middle initial, last name) Neal Tetervin			
6. REPORT DATE 21 March 1972		7a. TOTAL NO. OF PAGES 161	7b. NO. OF REFS 20
8a. CONTRACT OR GRANT NO		9a. ORIGINATOR'S REPORT NUMBER(S) NOLTR 72-80	
b. PROJECT NO. ORD Task 35A-001-100-31			
c.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.			
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Naval Ordnance Systems Command Washington, D.C. 20360	
13. ABSTRACT The Magnus force is found by adding the boundary layer displacement surface to the body and then calculating the force on the resulting non-axisymmetric body. The displacement surface is calculated by the simultaneous integration of a differential equation for a streamline, two partial-differential boundary layer integral-momentum equations with the boundary layer thickness and the tangent of the angle between the local direction of the surface friction and potential flow streamline as dependent variables, and a partial differential equation for the displacement surface thickness. The method is applied to a half-ellipsoid of fineness ratio 5 at 4 degrees angle of attack. Slender-body theory is used to calculate the force.			

DD FORM 1 NOV 65 1473

(PAGE 1)

S/N 0101-807-6801

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Security Classification

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CALCULATION OF MAGNUS FORCES ON AXISYMMETRIC BODIES AT SMALL ANGLES
OF ATTACK WITH INCOMPRESSIBLE TURBULENT BOUNDARY LAYERS

This report presents the derivation of a method for calculating the Magnus force and moment on a spinning body of revolution. For an example the force and moment on a fineness ratio 5 half-ellipsoid at an angle of attack of 4 degrees with a non-dimensional spin speed of .25 are calculated.

Appreciation is expressed to Mrs. Carolyn Piper and to Mrs. Rita Bell who programmed the method on an electronic computer and conducted the calculations. This work was supported by the Naval Ordnance Systems Command, under ORD Task 35A-001-100-31.

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NOMENCLATURE

\hat{a}	unit vector parallel to body surface in plane $\theta = \text{constant}$, positive in direction of increasing x
a	angle from meridian $\theta = \text{constant}$ to tangent to potential flow streamline, positive in direction of increasing θ
a_0	constant, Equation (40)
A	$q_\infty - \omega r_0 \sin a$
dA	element of area on resultant body, body plus displacement surface
\hat{b}	$\hat{c} \times \hat{a}$, unit vector parallel to body surface in plane $x = \text{constant}$
b_0	constant, Equation (40)
B	$\omega r_0 \sin a$
\hat{c}	unit vector normal to body surface, outward from surface
c_1	constant, Equation (79)
c_2	constant, Equation (80)
c_3	c_2/c_1
c_4	constant, Equation (83)
C	$\omega r_0 \cos a$
C_D	drag coefficient based on area $\pi \bar{r}_M^2$
C_n	Magnus yawing moment coefficient (Equation (154))
C_N	normal force coefficient (Equation (152))
C_p	pressure coefficient $\frac{\bar{p} - \bar{p}_\infty}{\frac{\rho_\infty}{2} \bar{V}_\infty^2}$
C_Y	Magnus side force coefficient (Equation (153))
d	displacement of origin of x, r, θ system from origin of ξ, h, β system (Figure 8)

D_1	$\sqrt{\Delta s_x^2 + \left(\frac{\partial \delta^*}{\partial s_x} \Delta s_x\right)^2}$
D_2	$\sqrt{r_0^2 \Delta \theta^2 + \left(\frac{\partial \delta^*}{\partial \theta} \Delta \theta\right)^2}$
\hat{e}_1	unit vector in direction of free stream velocity (Figure 8)
\hat{e}_2	unit vector normal to \hat{e}_1 and in ξ, x plane (Figure 8)
\hat{e}_3	$\hat{e}_1 \times \hat{e}_2$ (Figure 8)
\bar{F}	$\bar{F}_Y + i \bar{F}_Z$, complex Magnus force on portion of body between cross section $\xi = C$ and body nose
\bar{F}_Y	component of Magnus force \bar{F} along Y axis (Figure 6)
\bar{F}_Z	component of Magnus force \bar{F} along Z axis (Figure 6)
G_i	function G_i (See Table I)
h	radial distance in plane $\xi = \text{constant}$ (Figure 8)
h_0	radial distance to point on body in plane $\xi = \text{constant}$ (Figure 6)
\hat{h}	unit radial vector in plane $\xi = \text{constant}$ (Figure 8)
\hat{i}	unit vector along body axis (Figure 8)
i	$\sqrt{-1}$
I	radius of starting circle, (Equation (100))
\hat{j}	unit vector normal to \hat{i} and in x, ξ plane (Figure 8)
J_i	parameter (See Table III)
k	constant in friction formula (Equation (95))
K_1, K_2	parameters for ellipsoid (Equations (150) and (151))
l	exponent (Equation (40))
\bar{L}	reference length
m	exponent in friction formula (Equation (95))
M	Mach number
\bar{M}_Y	Magnus moment about nose of body acting on portion of body between cross section $\xi = C$ and the body nose, positive as shown in Figure 1

n	exponent in velocity profile formula (Equation (52))
\hat{n}	unit vector normal to body plus displacement surface, outward from surface
N	number of base radii behind base beyond which wake thickness is constant and equal to R_W^*
\vec{N}	vector normal to body plus displacement surface, outward from surface
p	spin rate parameter $\frac{\bar{\omega}^2 M}{\bar{V}_\infty}$
P	static pressure
q_e	velocity at outer edge of boundary layer
Q	right hand side of Equation (24)
\vec{Q}	velocity vector (Equation (105))
r	radial distance in plane $x = C$ (Figure 8)
r_o	radius of body of revolution
r_M	maximum radius of body of revolution
r'	radial coordinate of section of resultant body in Z' plane (Figure 7)
\hat{r}	unit vector along radial direction in $x = \text{constant}$ plane (Figure 8)
R	radial coordinate of resultant body, $r_o + \delta^*$
Re_L	reference Reynolds number $\frac{\bar{V}_\infty \bar{L}}{\bar{\nu}}$
R_W	displacement radius of wake (Equation (C-1))
R_W^*	constant displacement radius of wake far behind body (Equation (C-1))
s	distance on body surface along potential flow streamline
s_x	distance along body surface in plane $\theta = \text{constant}$
s_θ	distance along body surface in plane $x = \text{constant}$
s_ξ	distance along section of body plus displacement surface in plane $\xi = \text{constant}$ (Figure 7)

\vec{S}	vector from origin of ξ, η, β system of coordinates (Figure 8)
t	thickness ratio, minor axis divided by major axis for ellipsoid
T	temperature
u	velocity parallel to surface and in plane $\theta = \text{constant}$
$u_{e, s}$	value of $\frac{du_e}{ds_x}$ at stagnation point
u_q	velocity parallel to surface and in direction of q_e
v	velocity parallel to surface and in plane $x = \text{constant}$
v_q	velocity parallel to surface and normal to direction of q_e
V_s	velocity along circle in Z plane (Figure 7)
V'_s	velocity along cross section in Z' and z_0 planes (Figure 7)
V_n	velocity normal to circle in Z plane (Figure 7)
V'_n	velocity normal to cross section in Z' and z_0 planes (Figure 7)
\bar{V}_∞	magnitude of free stream velocity
\vec{V}_e	velocity vector along displacement surface (Equation (10))
w	velocity normal to body surface
x	distance along axis of revolution
x_0	value of x at intersection of axis of revolution and plane $\xi = C$
y	distance from body surface in direction normal to surface
y_q	value of y slightly larger than δ
Y	coordinate in z_0 plane (Figure 6)
z_0	complex plane coinciding with plane $\xi = \text{constant}$ and having origin of coordinates on ξ axis, $z_0 = Y + i Z$ (Figure 6, 7)
Z'	complex plane coinciding with plane $\xi = \text{constant}$ and having origin of coordinates on x axis (Figure 7)

Z	complex plane in which section of body plus displacement surface is a circle as a result of a transformation (Figure 7)
α	angle of attack of body
β_0	$\tan^{-1} \psi$
β	angular coordinate in $\xi = \text{constant}$ plane (Figures 6, 7, 8)
$\hat{\beta}$	unit vector in direction of increasing β (Figures 6, 8)
γ	angular coordinate in Z' plane (Figure 7)
Δ	integration interval 2π divided into equal steps of length Δ (Equation (144))
δ	boundary layer thickness
δ^*	boundary layer displacement thickness defined by Equation (23)
∇	gradient symbol
ϵ	$(\sigma - \gamma)$ see Equation (128)
ζ	y/δ
η	starting angle (Equation (100))
$\hat{\theta}$	unit vector normal to \hat{r} and in direction of increasing θ (Figure 8) ($\hat{\theta} = \hat{r} \times \hat{i}$)
θ	angular coordinate (Figure 2)
κ	ratio of specific heat at constant pressure to specific heat at constant volume
λ	radial coordinate in circle plane (Figure 7)
$\bar{\mu}$	viscosity
$\bar{\nu}$	kinematic viscosity $\bar{\mu}/\bar{\rho}$
$\hat{\nu}$	unit vector normal to section of body $\xi = \text{constant}$ (Figures 6, 7)
ξ	distance in direction of free stream velocity (Figures 6, 8)
$\bar{\rho}$	density
σ	angle in Z plane (Figure 7)

τ_w	wall shear stress
τ_{wx}	wall shear stress along $\theta = \text{constant}$ direction
$\tau_{w\theta}$	wall shear stress along $x = \text{constant}$ direction
ϕ	angle between surface of body and x axis in plane $\theta = \text{constant}$ (Figure 2)
Φ	three-dimensional perturbation velocity potential (Equation (105))
Φ_0	two-dimensional perturbation velocity potential in plane $\xi = c$
χ_1	angle defined by Equations (12) and (13)
χ_2	angle defined by Equations (14) and (15)
ψ	tangent of angle measured from direction of potential flow streamline to direction of surface shear stress, positive in direction of increasing θ
ω	angular spin velocity of body $\frac{\bar{\omega} \bar{L}}{\bar{V}_\infty}$, positive as shown in (Figure 1)
Ω	defined by $r' = r_0 e^\Omega$ (Equation (124))
Ω_0	defined by $\lambda = r_0 e^{\Omega_0}$ (Equation (125))

Subscripts

b	at base of body
e	at outer edge of boundary layer
M	maximum value
s	at stagnation point
w	at surface
W	wake
δ^*	on displacement thickness surface
∞	very far ahead of body

Superscripts

"	fluctuating quantity
< >	mean value
*	value very far behind base
—	dimensional quantity

INTRODUCTION

When a spinning body of revolution is flying at an angle of attack, a force normal to the angle of attack plane acts on the body. Associated with this side force is a moment. The force and moment are known as the Magnus force and moment. Although the Magnus force is usually only a small fraction of the normal force it, and its moment about the body center of gravity, can have an important effect on the body trajectory. In order to predict the Magnus force and moment and in order to better understand experimental results a theory is needed.

The problem is to calculate the Magnus force and moment given only the shape of a body of revolution, its speed along the flight path, its spin rate, its angle of attack, and the properties of the atmosphere. If the flow were symmetric about the plane formed by the axis of the body and the free-stream velocity vector there would be no side force, a force perpendicular to the plane of symmetry. There would also be no yawing moment. Because, however, the fluid through which the body is moving is viscous, a boundary layer is present on the body. The spin of the body combined with the angle of attack causes the boundary layer to be unsymmetric with respect to the plane of symmetry, the plane formed by the axis of revolution and the free-stream velocity vector. The resultant configuration is shown in Figure 1. Because the symmetry of the flow about the angle of attack plane is destroyed by the unsymmetric boundary layer a side force and moment exist. For the symmetry to be destroyed, both spin and angle of attack must be present.

In the present investigation the Magnus force and moment are calculated for a body with no separation of the boundary layer. The boundary layer is therefore thin over the entire body and there are no regions of vorticity shed into the flow ahead of the body base. To calculate the Magnus force and moment for such a flow the inviscid flow around the body is found first. Then the boundary layer displacement thickness surface surrounding the body is calculated by use of boundary layer theory. The displacement thickness surface is added to the body and the force and moment calculated for the resultant body in an inviscid flow.

This method for the calculation of the Magnus force and moment is that of Martin (Reference 1) who presents a theory for the Magnus force and moment on a cylinder at a small angle of attack. The

boundary layer is laminar and incompressible. Quantities of higher order than the second in angle of attack, spin velocity, and distance from the leading edge and their products with one another are neglected. Martin finds that the Magnus force is directly proportional to the product of the angle of attack, spin velocity ratio, body length, and the displacement thickness at the body base at zero angle of attack. For turbulent flow Martin replaces a calculated constant of proportionality by an unknown coefficient.

Platou (Reference 2) presents experimental data for supersonic flow and concludes that Martin's incompressible flow theory predicts the correct order of magnitude for 3 to 5 caliber bodies with laminar boundary layers. To apply Martin's theory to a bullet shaped body an arbitrary allowance must be made for the nose portion of non-constant diameter. The extrapolation of Martin's theory to turbulent flow correctly predicts the Magnus force to be directly proportional to the product of angle of attack and spin rate. Platou also finds that the theory of Kelly and Thacker (Reference 3), which includes a radial pressure gradient and skin friction effect, does not agree with Martin's prediction nor does it agree with the experimental result that the Magnus force depends on the spin to the first power.

Sedney (Reference 4) calculates the Magnus force and moment on a slender spinning cone at a small angle of attack in supersonic flow. The boundary layer flow is laminar. Terms of higher order than the first in angle of attack, spin velocity, and in the product of spin velocity and angle of attack are neglected. The method follows Martin and also, like Martin, Sedney uses slender-body theory to calculate the force and moment on the body that results when the boundary layer displacement surface is added to the body of revolution. No experimental test of the predictions seems to be available.

In the present investigation a method is developed to calculate the Magnus force and moment for a body of revolution of general shape with an unseparated turbulent boundary layer. Although the boundary layer calculation is for incompressible flow, the calculation method can be used for Mach numbers up to the transonic range because Mach number effects on boundary layer flow are usually small for local Mach numbers less than unity. The boundary layer calculation is based on the momentum integral method. The force on the body is calculated by slender-body theory (References 5 and 6). A discussion is given of the calculation of the force by a more exact method than slender-body theory, namely, the method of Hess and Smith (References 7 and 8).

ANAYLSIS

In order to calculate the force and moment on a spinning body the effective shape of the body in an inviscid flow is needed. The effective shape is found by calculating the boundary layer displacement surface and adding it to the body of revolution. The displacement surface δ^* is the surface, which, added to a body in an inviscid flow, results in a body with the same streamlines as those outside

the boundary layer in the real viscous flow (Reference 9). Therefore, because according to boundary layer theory the pressure difference across the boundary layer is negligible, the pressure on the real body in the viscous flow is the same as on the body plus δ^* in the inviscid flow.

Displacement Surface

To find the displacement surface, δ^* , use is made of the fact that the streamlines outside the boundary layer are not changed by replacing the body with the boundary layer over it in the viscous flow by the body with δ^* added to it in an inviscid flow. Consequently the velocity component w on these streamlines is also unchanged. To find w on a streamline near the outer edge of the boundary layer the continuity equation is used; for a steady flow with the coordinate system shown in Figure 2 it is

$$\cos\phi \frac{\partial}{\partial x} (\rho u r) + \frac{\partial}{\partial \theta} (\rho v) + \frac{\partial}{\partial y} (\rho w r) = 0 \quad (1)$$

where ρ , u , and v are time mean values and ρw is $(\rho w + \langle \rho w \rangle)$. Equation (1) is the same as Equation (19) page 414 Reference 10 when r is put equal to r_0 . The coordinate system is fixed in the fluid and the body rotates around its axis of symmetry. All quantities are non-dimensional; the velocities are non-dimensionalized by \bar{V}_∞ , the lengths by \bar{L} , the densities by $\bar{\rho}_\infty$, and the pressures and shear stresses by $\bar{\rho}_\infty \bar{V}_\infty^2$.

An integration of (1) with respect to y up to y_q , where y_q is slightly greater than the boundary layer thickness, results in

$$(\rho w)_{y_q} = - \frac{\cos\phi}{r} \int_0^{y_q} \frac{\partial}{\partial x} (\rho u r) dy - \frac{1}{r} \int_0^{y_q} \frac{\partial}{\partial \theta} (\rho v) dy \quad (2)$$

For an inviscid flow over the body with δ^* added to it the result for $(\rho w)_{y_q}$ is

$$(\rho w)_{y_q} = - \frac{\cos \phi}{r} \int_{\delta^*}^{y_q} \frac{\partial}{\partial x} (\rho_e u_e r) dy - \frac{1}{r} \int_{\delta^*}^{y_q} \frac{\partial}{\partial \theta} (\rho_e v_e) dy + (\rho w)_{\delta^*} \quad (3)$$

The term $(\rho w)_{\delta^*}$ appears because the δ^* surface has a slope with respect to the body and w is normal to the surface without δ^* . In Equation (3) ρ , u , and v have the subscript "e" because the flow is inviscid and the velocity varies negligibly slowly with y .

Then, because the streamline at y_q is the same for both the viscous and the inviscid flow ρw is also the same. Therefore, equating (2) and (3) the result is

$$\begin{aligned} & - \frac{\cos \phi}{r} \int_0^{y_q} \frac{\partial}{\partial x} [r(\rho_e u_e - \rho u)] dy - \frac{1}{r} \int_0^{y_q} \frac{\partial}{\partial \theta} (\rho_e v_e - \rho v) dy \\ & + \frac{\cos \phi}{r} \int_0^{\delta^*} \frac{\partial}{\partial x} (\rho_e u_e r) dy + \frac{1}{r} \int_0^{\delta^*} \frac{\partial}{\partial \theta} (\rho_e v_e) dy + (\rho w)_{\delta^*} = 0 \end{aligned} \quad (4)$$

The relations

$$\frac{\partial}{\partial x} \int_0^{\delta^*} \rho_e u_e r dy = \int_0^{\delta^*} \frac{\partial}{\partial x} (\rho_e u_e r) dy + \frac{\partial \delta^*}{\partial x} r \rho_e u_e \quad (5)$$

$$\frac{\partial}{\partial \theta} \int_0^{\delta^*} \rho_e v_e dy = \int_0^{\delta^*} \frac{\partial}{\partial \theta} (\rho_e v_e) dy + \frac{\partial \delta^*}{\partial \theta} \rho_e v_e \quad (6)$$

$$\frac{\partial}{\partial x} \int_0^{y_q} r (\rho_e u_e - \rho u) dy = \int_0^{y_q} \frac{\partial}{\partial x} [r (\rho_e u_e - \rho u)] dy \quad (7)$$

$$\frac{\partial}{\partial \theta} \int_0^{y_q} (\rho_e v_e - \rho v) dy = \int_0^{y_q} \frac{\partial}{\partial \theta} (\rho_e v_e - \rho v) dy \quad (8)$$

are now used in Equation (4). The distance r is taken as the body radius r_o , consistent with the thin boundary layer assumption. In the integrals in Equations (5), (6), (7), and (8), $\rho_e u_e$ and $\rho_e v_e$ are equal to their values at y_q . This causes an error of $O(\delta^2)$ in the integrals, which is negligible with respect to the integrals themselves which are of $O(\delta)$. Then Equation (4) becomes

$$\begin{aligned}
 & - \frac{\cos \phi}{r_o} \frac{\partial}{\partial x} r_o \int_0^{y_q} (\rho_e u_e - \rho u) dy - \frac{1}{r_o} \frac{\partial}{\partial \theta} \int_0^{y_q} (\rho_e v_e - \rho v) dy + \frac{\delta^*}{r_o} \cos \phi \frac{\partial}{\partial x} (r_o \rho_e u_e) \\
 & + \frac{\delta^*}{r_o} \frac{\partial}{\partial \theta} (\rho_e v_e) + (\rho w)_{\delta^*} = 0
 \end{aligned} \tag{9}$$

To find $(\rho w)_{\delta^*}$ use is made of the fact that in the inviscid flow over the body with δ^* added to it the velocity component normal to the displacement surface δ^* is zero. The velocity vector \vec{V}_e along the δ^* surface is

$$\vec{V}_e = \hat{a} u_e + \hat{b} v_e + \hat{c} w_e \tag{10}$$

where \hat{a} is a unit vector along the body surface and in the x direction, \hat{b} is a unit vector along the body surface in a plane $x = \text{constant}$, and \hat{c} is a unit vector normal to the body surface and outward. The condition for no velocity normal to the displacement surface is

$$\vec{V}_e \cdot \vec{N} = 0 \tag{11}$$

where \vec{N} is a vector normal to the body plus δ^* and outward from the surface. To find \vec{N} note that \vec{N} is normal to each of the two vectors $(\hat{a} \cos \chi_1 + \hat{c} \sin \chi_1)$ and $(\hat{b} \cos \chi_2 + \hat{c} \sin \chi_2)$ which lie in the displacement surface along $\theta = c$ and $x = c$, respectively. The angles χ_1 and χ_2 are given by

$$\cos \chi_1 = \frac{\Delta s_x}{D_1} \tag{12}$$

$$\sin \chi_1 = \frac{\frac{\partial \delta^*}{\partial s_x} \Delta s_x}{D_1} \quad (13)$$

$$\cos \chi_2 = \frac{r_o \Delta \theta}{D_2} \quad (14)$$

$$\sin \chi_2 = \frac{\frac{\partial \delta^*}{\partial \theta} \Delta \theta}{D_2} \quad (15)$$

where

$$D_1 = \sqrt{\Delta s_x^2 + \left(\frac{\partial \delta^*}{\partial s_x} \Delta s_x \right)^2} \quad (16)$$

and

$$D_2 = \sqrt{r_o^2 \Delta \theta^2 + \left(\frac{\partial \delta^*}{\partial \theta} \Delta \theta \right)^2} \quad (17)$$

Then

$$\vec{N} = (\hat{a} \cos \chi_1 + \hat{c} \sin \chi_1) \times (\hat{b} \cos \chi_2 + \hat{c} \sin \chi_2) \quad (18)$$

or

$$\vec{N} = \hat{c}\cos\chi_1\cos\chi_2 - \hat{b}\cos\chi_1\sin\chi_2 - \hat{a}\sin\chi_1\cos\chi_2 \quad (19)$$

Then (11) becomes

$$\vec{V}_e \cdot \vec{N} = w_e \cos\chi_1 \cos\chi_2 - v_e \cos\chi_1 \sin\chi_2 - u_e \sin\chi_1 \cos\chi_2 = 0 \quad (20)$$

or with (12) through (17),

$$w_e = u_e \frac{\partial \delta^*}{\partial s_x} + \frac{v_e}{r_o} \frac{\partial \delta^*}{\partial \theta} \quad (21)$$

or

$$(\rho w)_{\delta^*} = \rho_e u_e \cos\phi \frac{\partial \delta^*}{\partial x} + \frac{\rho_e v_e}{r_o} \frac{\partial \delta^*}{\partial \theta} \quad (22)$$

When (22) is substituted into (9) the result, after rearranging terms, is

$$\cos\phi \frac{\partial}{\partial x} \left\{ r_o [\rho_e u_e \delta^* - \int_0^{y_q} (\rho_e u_e - \rho u) dy] \right\} + \frac{\partial}{\partial \theta} [\rho_e v_e \delta^* - \int_0^{y_q} (\rho_e v_e - \rho v) dy] = 0 \quad (23)$$

Equation (23) is the equation for the displacement surface δ^* (see Reference 9). The Equation (23) is a partial differential equation for δ^* . When Equation (23) is expanded the result is

$$\begin{aligned}
 r_0 \cos \phi \rho_e u_e \frac{\partial \delta^*}{\partial x} + \rho_e v_e \frac{\partial \delta^*}{\partial \theta} = & - \delta^* \left[r_0 \cos \phi \frac{\partial \rho_e u_e}{\partial x} + \cos \phi \frac{\partial r_0}{\partial x} \rho_e u_e + \frac{\partial \rho_e v_e}{\partial \theta} \right] \\
 & + \cos \phi \frac{\partial r_0}{\partial x} \int_0^{\delta} (\rho_e u_e - \rho u) dy + r_0 \cos \phi \frac{\partial}{\partial x} \int_0^{\delta} (\rho_e u_e - \rho u) dy \\
 & + \frac{\partial}{\partial \theta} \int_0^{\delta} (\rho_e v_e - \rho v) dy
 \end{aligned} \tag{24}$$

where the upper limit y_q in (23) has been replaced by δ in (24); δ is the smallest value of y for which simultaneously $\rho u = \rho_e u_e$ and $\rho v = \rho_e v_e$.

Equation (24) is a Lagrange linear partial differential equation. To obtain a solution the two subsidiary equations

$$\frac{dx}{r_0 \cos \phi \rho_e u_e} = \frac{d\theta}{\rho_e v_e} \tag{25}$$

and

$$\frac{dx}{r_0 \cos \phi \rho_e u_e} = \frac{d\delta^*}{Q} \tag{26}$$

are to be integrated; the quantity Q is the right hand side of Equation (24). Equation (25) is

$$\frac{d\theta}{dx} = \frac{v_e}{r_o u_e \cos\phi} \quad (27)$$

and is the equation of an inviscid flow streamline over the body. To integrate (26) requires that the integrals in (24) be known. That is, the boundary layer velocity distribution and thickness must be known. When the boundary layer properties are known, Equations (25) and (26) are integrated together thereby giving δ^* along a streamline. By integrating (25) and (26) along a sufficient number of streamlines the displacement surface δ^* on the body is calculated.

Momentum Integral Equations

For thin boundary layers the equations of motion are (Reference 10 page 14)

$$\rho u \cos\phi \frac{\partial u}{\partial x} + \frac{\rho v}{r_o} \frac{\partial u}{\partial \theta} + \rho w \frac{\partial u}{\partial y} - \frac{\cos\phi}{r_o} \frac{\partial r_o}{\partial x} \rho v^2 = - \cos\phi \frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y} \quad (28)$$

and

$$\rho u \cos\phi \frac{\partial v}{\partial x} + \frac{\rho v}{r_o} \frac{\partial v}{\partial \theta} + \rho w \frac{\partial v}{\partial y} + \frac{\cos\phi}{r_o} \frac{\partial r_o}{\partial x} \rho u v = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial \tau_\theta}{\partial y} \quad (29)$$

and the continuity equation is

$$\frac{\cos\phi}{r_o} \frac{\partial}{\partial x} (\rho u r_o) + \frac{1}{r_o} \frac{\partial}{\partial \theta} (\rho v) + \frac{\partial}{\partial y} (\rho w) = 0 \quad (1a)$$

To obtain the momentum integral equations the x equation of motion (28) and the θ equation of motion (29) are integrated with respect to y in the usual way with the help of the equation of continuity. The result for the x equation is

$$\begin{aligned}
 & r_o \cos \phi \frac{\partial}{\partial x} \int_0^\delta \rho u (u_e - u) dy + \sin \phi \int_0^\delta \rho u (u_e - u) dy + r_o \cos \phi \frac{\partial u_e}{\partial x} \int_0^\delta (\rho_e u_e - \rho u) dy \\
 & + \frac{\partial}{\partial \theta} \int_0^\delta \rho v (u_e - u) dy + \frac{\partial u_e}{\partial \theta} \int_0^\delta (\rho_e v_e - \rho v) dy - \sin \phi \int_0^\delta (\rho_e v_e^2 - \rho v^2) dy = \tau_{w_x} r_o
 \end{aligned} \tag{30}$$

For the θ equation the result is

$$\begin{aligned}
 & \frac{\partial}{\partial \theta} \int_0^\delta \rho v (v_e - v) dy + \frac{\partial v_e}{\partial \theta} \int_0^\delta (\rho_e v_e - \rho v) dy + r_o \cos \phi \frac{\partial}{\partial x} \int_0^\delta \rho u (v_e - v) dy \\
 & + \sin \phi \int_0^\delta \rho u (v_e - v) dy + r_o \cos \phi \frac{\partial v_e}{\partial x} \int_0^\delta (\rho_e u_e - \rho u) dy + \sin \phi \int_0^\delta (\rho_e u_e v_e - \rho u v) dy \\
 & = \tau_{w_\theta} r_o
 \end{aligned} \tag{31}$$

These equations are the same as on page 416 of Reference 10 in different notation. In order to use the momentum Equations (30) and (31) to calculate the terms $\int_0^\delta (\rho_e u_e - \rho u) dy$ and $\int_0^\delta (\rho_e v_e - \rho v) dy$

in Equation (24) for the displacement surface δ^* an expression for the velocity profile through the boundary layer is needed.

Velocity Profile

The velocity profile of the velocity component in the direction of the outer streamline is assumed to be

$$\frac{u_q - u_{q,w}}{q_e - u_{q,w}} = f(\zeta) \quad (32)$$

where $\zeta = y/\delta$, and the velocity profile for the velocity component perpendicular to the outer streamline is assumed to be

$$\frac{v_q - v_{q,w}}{q_e - u_{q,w}} = \psi g(\zeta) f(\zeta) \quad (33)$$

In (32) and (33) q_e is the speed along the streamline at the outer edge of the boundary layer; u_q is the speed in the direction of the outer streamline and v_q is the speed in the direction perpendicular to the outer streamline. The quantity ψ is an as yet undetermined parameter. At the surface (see Figure 3)

$$u_{q,w} = \omega r_o \sin a, \quad (34)$$

and

$$v_{q,w} = \omega r_o \cos a, \quad (35)$$

where a is the angle between the outer streamline and a line $\theta = c$. Also, from (32) $f(0) = 0$ and $f(1) = 1$. At $\zeta = 1$, $v_q = 0$ by definition. Therefore from (33) and $f(1) = 1$ it follows that

$$g(1) = \frac{-v_{q,w}}{\psi(q_e - u_{q,w})} \quad (36)$$

Also, division of (33) by (32) results in

$$\frac{v_q - v_{q,w}}{u_q - u_{q,w}} = \psi q \quad (37)$$

As $y \rightarrow 0$ (37) becomes

$$\lim_{y \rightarrow 0} \frac{v_q - v_{q,w}}{u_q - u_{q,w}} = \psi g(0) \quad (38)$$

The quantity $g(0)$ is put equal to unity. Then (38) becomes

$$\lim_{y \rightarrow 0} \frac{\partial v_q}{\partial u_q} = \lim_{y \rightarrow 0} \frac{\frac{\partial v_q}{\partial y}}{\frac{\partial u_q}{\partial y}} = \psi \quad (39)$$

The quantity $\bar{\mu}_w \left(\frac{\partial \bar{v}_q}{\partial \bar{y}} \right)_w$ equals the shear on the surface in a direction normal to the outer streamline; a positive value means the shear acts in the direction of increasing θ . The quantity $\bar{\mu}_w \left(\frac{\partial \bar{u}_q}{\partial \bar{y}} \right)_w$ is equal to the shear acting on the surface in the direction of the outer streamline flow. Consequently ψ is the tangent of the angle measured from the direction of the outer streamline to the direction of the surface shear stress; ψ is positive in the direction of increasing θ .

The function $g(\zeta)$ is taken to be

$$g(\zeta) = (1-\zeta)^k + a_0 + b_0 \zeta \quad (40)$$

The condition $g(0)=1$, previously imposed, gives $a_0=0$. From the expression (36) for $g(1)$ and from (40), the result is

$$b_0 = - \frac{v_{q,w}}{\psi[q_e - u_{q,w}]}$$

Therefore (40) becomes

$$g(\zeta) = (1-\zeta)^l - \frac{v_{q,w}}{\psi[q_e - u_{q,w}]} \zeta \quad (41)$$

The exponent l is found by considering conditions at the stagnation point of a rotating body at an angle of attack; thus the stagnation point is not on the axis of rotation. At the stagnation point the outer stream velocity is zero and the surface has a rotational velocity ωr_0 . Let

$$v = \omega r_0 F(\zeta) \quad (42)$$

where $F(0)=1$ and $F(1)=0$, that is, the fluid has velocity ωr_0 at the surface and zero velocity outside the boundary layer. Also, u , the velocity in the direction of s_x is zero at the stagnation point. The relations between the velocity components u_q and v_q and the velocity components u and v are

$$u_q = u \cos \alpha + v \sin \alpha \quad (43)$$

and

$$v_q = -u \sin \alpha + v \cos \alpha \quad (44)$$

At the stagnation point (43) and (44) becomes, with (42),

$$u_q = \omega r_o F \sin a \quad (45)$$

and

$$v_q = \omega r_o F \cos a \quad (46)$$

Equations (45), (34), (32) and $q_e=0$ result in

$$F = 1-f \quad (47)$$

From (34), (35), $q_e=0$, (46) and (47) are used in (33) the result is

$$\cot a = \psi g \quad (48)$$

From (39), (45) and (46) it follows that

$$\cot a = \psi \quad (49)$$

Therefore from (48) it follows that $g=1$ for all ζ at the stagnation point. Now consider Equation (41). At the stagnation point $q_e=0$, $\psi=\cot a$, and relations (45) and (46) with $F(o)=1$ hold.

Equation (41) then becomes

$$g(\zeta) = (1-\zeta)^k + \zeta \quad (50)$$

Therefore to have $g(\zeta)=1$ at the stagnation point, put $l=1$.
Then (41) becomes

$$g(\zeta) = 1 - \zeta \left\{ 1 + \frac{v_{q,w}}{[q_e^{-u_{q,w}}]} \right\} \quad (51)$$

The velocity profile (32) is taken to be a power profile by putting

$$f(\zeta) = \zeta^n \quad (52)$$

Because the velocities that appear in the two momentum integral Equations (30) and (31) contain u and v it is necessary to express u and v in terms of the velocities u_q and v_q that appear in (32) and (33). The relations are

$$u = u_q \cos \alpha - v_q \sin \alpha \quad (53)$$

and

$$v = u_q \sin \alpha + v_q \cos \alpha \quad (54)$$

When u_q and v_q are found from (32) and (33) and (34), (35), (51), and (52) are used, Equations (53) and (54) become

$$u = (A\zeta^n + B) \cos \alpha - [A\psi(\zeta^n - \zeta^{n+1}) + C(1 - \zeta^{n+1})] \sin \alpha \quad (55)$$

and

$$v = (A\zeta^n + B)\sin\alpha + [A\psi(\zeta^n - \zeta^{n+1}) + C(1 - \zeta^{n+1})]\cos\alpha \quad (56)$$

where

$$A = q_e - u_{q,w} = q_e - \omega r_o \sin\alpha \quad (57)$$

$$B = u_{q,w} = \omega r_o \sin\alpha \quad (58)$$

$$C = v_{q,w} = \omega r_o \cos\alpha \quad (59)$$

Differential Equations for δ and ψ

Expressions (55) and (56) are put into the integral momentum Equations (30) and (31) and the integrals evaluated. The density is taken constant from here on in the boundary layer analysis. A sample term is $\int_0^\delta (u_e - u) dy$ which is written as $\delta \int_0^1 (u_e - u) d\zeta$

The result is

$$\int_0^1 (u_e - u) d\zeta = A \frac{n}{n+1} \cos\alpha - A\psi \frac{n}{n+1} \sin\alpha + (A\psi + C) \frac{n+1}{n+2} \sin\alpha$$

or

$$\int_0^1 (u_e - u) d\zeta = AG_7 \cos\alpha + \psi AG_2 \sin\alpha + CG_4 \sin\alpha$$

where

$$G_2 = \frac{1}{(n+1)(n+2)}$$

$$G_4 = \frac{n+1}{n+2}$$

$$G_7 = \frac{n}{n+1}$$

There are fourteen G's; they are listed in Table 1.

In Table II are listed the integrals that occur in Equations (30) and (31). The procedure results in two partial differential equations with δ and ψ as dependent variables and x and θ as independent variables. The x equation, the result of (30) and (55) and (56) is

$$\begin{aligned} & r_0 \cos \phi \frac{\partial \delta}{\partial x} [\psi^2 J_1 + \psi J_2 + J_3] + r_0 \cos \phi \frac{\partial \psi}{\partial x} \delta [2\psi J_1 + J_2] \\ & + \frac{\partial \delta}{\partial \theta} [\psi^2 J_6 + \psi J_7 + J_8] + \frac{\partial \psi}{\partial \theta} \delta [2\psi J_6 + J_7] \\ & = r_0 \tau_{w_x} - \delta \left\{ \left[\psi^2 \frac{\partial J_1}{\partial x} + \psi \frac{\partial J_2}{\partial x} + \frac{\partial J_3}{\partial x} \right] r_0 \cos \phi \right. \\ & \quad + [\psi^2 J_1 + \psi J_2 + J_3] \frac{\partial r_0}{\partial x} \cos \phi + r_0 \cos \phi \frac{\partial u_e}{\partial x} [\psi J_4 + J_5] \\ & \quad + \left[\psi^2 \frac{\partial J_6}{\partial \theta} + \psi \frac{\partial J_7}{\partial \theta} + \frac{\partial J_8}{\partial \theta} \right] + \frac{\partial u_e}{\partial \theta} [\psi J_{18} + J_{19}] \\ & \quad \left. - [\psi^2 J_9 + \psi J_{10} + J_{11}] \frac{\partial r_0}{\partial x} \cos \phi \right\} \end{aligned} \tag{60}$$

The θ equation, the result of (31) and (55) and (56) is

$$\begin{aligned}
 & r_o \cos \phi \frac{\partial \delta}{\partial x} [\psi^2 J_6 + \psi J_{14} + J_{15}] + r_o \cos \phi \frac{\partial \psi}{\partial x} \delta [2\psi J_6 + J_{14}] \\
 & + \frac{\partial \delta}{\partial \theta} [\psi^2 J_9 + \psi J_{12} + J_{13}] + \frac{\partial \psi}{\partial \theta} \delta [2\psi J_9 + J_{12}] \quad (61) \\
 & = r_o \tau_{w_0} - \delta \{ [\psi^2 \frac{\partial J_6}{\partial x} + \psi \frac{\partial J_{14}}{\partial x} + \frac{\partial J_{15}}{\partial x}] r_o \cos \phi + [\psi^2 J_6 + \psi J_{14} + J_{15}] \frac{\partial r_o}{\partial x} \cos \phi \\
 & + \frac{\partial v_e}{\partial x} [\psi J_4 + J_5] r_o \cos \phi + [\psi^2 \frac{\partial J_9}{\partial \theta} + \psi \frac{\partial J_{12}}{\partial \theta} + \frac{\partial J_{13}}{\partial \theta}] \\
 & + \frac{\partial v_e}{\partial \theta} [\psi J_{18} + J_{19}] + [\psi^2 J_6 + \psi J_{16} + J_{17}] \frac{\partial r_o}{\partial x} \cos \phi \}
 \end{aligned}$$

The J 's are listed in Table III. They are independent of δ and ψ and are calculated from the inviscid flow velocity distribution over the body, spin rate, and the distribution $r_o(x)$.

The simultaneous solution of (60) and (61) gives $\frac{\partial \delta}{\partial x}$, $\frac{\partial \delta}{\partial \theta}$, $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial \theta}$ at a point x, θ . An iteration process is used. First (60) and (61) are solved for $(\frac{\partial \delta}{\partial x})_1$ and $(\frac{\partial \psi}{\partial x})_1$ with $\frac{\partial \delta}{\partial \theta} = 0$ and $\frac{\partial \psi}{\partial \theta} = 0$. Then the values of $(\frac{\partial \delta}{\partial x})_1$ and $(\frac{\partial \psi}{\partial x})_1$ are substituted into (60) and (61), and (60) and (61) are solved for $(\frac{\partial \delta}{\partial \theta})_1$ and $(\frac{\partial \psi}{\partial \theta})_1$. Then $(\frac{\partial \delta}{\partial \theta})_1$ and $(\frac{\partial \psi}{\partial \theta})_1$ are substituted into (60) and (61), and (60) and (61) are solved for $(\frac{\partial \delta}{\partial x})_2$ and $(\frac{\partial \psi}{\partial x})_2$. These values of $(\frac{\partial \delta}{\partial x})_2$ and $(\frac{\partial \psi}{\partial x})_2$ are substituted into (60) and (61), and (60) and (61) solved for $(\frac{\partial \delta}{\partial \theta})_2$ and $(\frac{\partial \psi}{\partial \theta})_2$. These values are then used to find $(\frac{\partial \delta}{\partial x})_3$ and $(\frac{\partial \psi}{\partial x})_3$ which are used to find $(\frac{\partial \delta}{\partial \theta})_3$ and $(\frac{\partial \psi}{\partial \theta})_3$ and so on until further iteration produces a negligible change in $(\frac{\partial \delta}{\partial x})$, $(\frac{\partial \delta}{\partial \theta})$, $(\frac{\partial \psi}{\partial x})$ and $(\frac{\partial \psi}{\partial \theta})$. In the calculations the

iteration is stopped either after 25 iterations or when $(\frac{\delta_i}{\delta_{i-1}} - 1)_{x+\Delta x}$ and $(\frac{\psi_i}{\psi_{i-1}} - 1)_{x+\Delta x}$ are both less than .001. The subscript i is the iteration number; $\delta_{x+\Delta x}$ and $\psi_{x+\Delta x}$ are the values of δ and ψ at the next point along a streamline. They are found from

$$\Delta\delta = \left[\left(\frac{\partial\delta}{\partial x_\theta} \right) + \left(\frac{\partial\delta}{\partial\theta} \right)_x \frac{d\theta}{dx} \right] \Delta x \quad (62)$$

and

$$\Delta\psi = \left[\left(\frac{\partial\psi}{\partial x_\theta} \right) + \left(\frac{\partial\psi}{\partial\theta} \right)_x \frac{d\theta}{dx} \right] \Delta x \quad (63)$$

where $\frac{d\theta}{dx}$ is obtained from Equation (27), the equation for a streamline.

Initial Conditions for δ and ψ

In order to begin the integration of (60) and (61), the values of δ and ψ at the stagnation point are needed. Up to now a method for calculating δ at the stagnation point of a spinning body when the stagnation point is not on the axis of rotation has not been found. Consequently, the needed value of δ is obtained by extrapolation from the value of δ at the stagnation point on a spinning body at zero angle of attack and the value of δ at the stagnation point of a non-spinning body at an angle of attack. Thus, it is assumed that δ at the stagnation point when spin and angle of attack are both present can be written as

$$\delta_{\alpha,p} = \delta_{0,0} + \left(\frac{\partial\delta}{\partial p} \right)_{0,0} p + \left(\frac{\partial\delta}{\partial\alpha} \right)_{0,0} \alpha \quad (64)$$

or with

$$\left(\frac{\partial \delta}{\partial p}\right)_{o,o} = \left(\frac{\delta_{o,p} - \delta_{o,o}}{p}\right)$$

and

$$\left(\frac{\partial \delta}{\partial \alpha}\right)_{o,o} = \left(\frac{\delta_{\alpha,o} - \delta_{o,o}}{\alpha}\right)$$

Equation (64) becomes

$$\delta_{\alpha,p} = \delta_{o,p} + \delta_{\alpha,o} - \delta_{o,o} \quad (65)$$

For $\alpha = 0$ the inviscid flow streamlines lie along $\theta = \text{constant}$ and the entire flow is independent of θ . Therefore to find $\delta_{o,p}$ and $\delta_{o,o}$ there is used, $a = 0$, $v_e = 0$, and $\frac{\partial}{\partial \theta} = 0$, in (60) and (61). As the stagnation point is approached $A \rightarrow q_e \rightarrow u_e$, $B \rightarrow 0$, and $c \rightarrow \omega r_o$. There is obtained, not putting $u_e = 0$, $r_o = 0$ yet,

$$\begin{aligned} J_1 &= 0 & J_5 &= u_e G_7 \\ J_2 &= 0 & J_6 &= 0 \\ J_3 &= u_e^2 G_7 G_8 & J_7 &= u_e^2 G_{14} \\ J_4 &= 0 & J_8 &= -u_e \omega r_o G_{10} \\ J_9 &= u_e^2 G_{11} & J_{15} &= -u_e \omega r_o G_9 \\ J_{10} &= -2u_e \omega r_o G_5 & J_{16} &= -u_e^2 G_{13} \end{aligned} \quad (66)$$

$$\begin{aligned}
 J_{11} &= \omega^2 r_o^2 G_6 & J_{17} &= -u_e \omega r_o G_9 \\
 J_{12} &= -2u_e \omega r_o G_5 & J_{18} &= -u_e G_2 \\
 J_{13} &= \omega^2 r_o^2 G_6 & J_{19} &= -\omega r_o G_4 \\
 J_{14} &= -u_e^2 G_{13}
 \end{aligned} \tag{66}$$

Because $a = 0$ for all x it can be shown that

$$\begin{aligned}
 \frac{\partial J_1}{\partial x} &= 0 & \frac{\partial J_6}{\partial x} &= 0 \\
 \frac{\partial J_2}{\partial x} &= 0 & \frac{\partial J_{14}}{\partial x} &= -2u_e \frac{\partial u_e}{\partial x} G_{13} \\
 \frac{\partial J_3}{\partial x} &= 2u_e \frac{\partial u_e}{\partial x} G_8 G_7 & \frac{\partial J_{15}}{\partial x} &= 0
 \end{aligned} \tag{67}$$

When these values are substituted into (60) and the resulting equation is divided through by $r_o u_e^2$ the result is

$$\begin{aligned}
 \frac{\partial \delta G_7 G_8}{\partial x} \cos \phi &= \frac{l_w x}{u_e^2} - \delta \left\{ \frac{1}{u_e} \frac{\partial u_e}{\partial x} \cos \phi (2G_7 G_8 + G_7) \right. \\
 &+ \frac{G_7 G_8}{r_o} \frac{\partial r_o}{\partial x} \cos \phi - \frac{\psi^2 G_{11}}{r_o} \frac{\partial r_o}{\partial x} \cos \phi + \frac{2\psi \omega}{u_e} G_5 \frac{\partial r_o}{\partial x} \cos \phi \\
 &\left. - \frac{\omega^2 r_o}{u_e^2} G_6 \frac{\partial r_o}{\partial x} \cos \phi \right\}
 \end{aligned} \tag{68}$$

At $\alpha = 0$ the relations $u_e = u'_{e,s} r_o$, and $r_o = s_x$ hold. Also

$\frac{\partial(\)}{\partial x} \cos \phi = \frac{\partial(\)}{\partial s_x}$. The friction coefficient is written as

$$\frac{\tau_{w_x}}{u_e^2} = \frac{\bar{\mu} \left(\frac{\partial \bar{u}}{\partial y} \right) w}{\bar{\rho} \bar{u}_e^2} = \frac{\bar{\nu} \left(\frac{\partial u/u_e}{\partial \zeta} \right) w}{\bar{V}_\infty u_e \delta \bar{L}} = \frac{\left(\frac{\partial u/u_e}{\partial \zeta} \right) w}{u'_{e,s} r_o \delta Re_L} \quad (69)$$

When these relations are substituted into (68), the result is

$$\begin{aligned} \frac{\partial \delta G_7 G_8}{\partial s} = \frac{1}{u'_{e,s} r_o \delta} \left\{ \frac{\left(\frac{\partial u/u_e}{\partial \zeta} \right) w}{Re_L} - u'_{e,s} \delta^2 [3G_7 G_8 + G_7 - \psi^2 G_{11} \right. \\ \left. + \frac{2\psi\omega}{u'_{e,s}} G_5 - \frac{\omega^2}{u'^2_{e,s}} G_6] \right\} \end{aligned} \quad (70)$$

The quantity $\delta G_7 G_8$ is equal to the ratio of the momentum thickness to the boundary layer thickness δ , $\int_0^1 \zeta^n (1-\zeta^n) d\zeta$. At the stagnation point $r_o = 0$. Consequently to avoid an infinite value of $\frac{\partial \delta G_7 G_8}{\partial s}$,

which is physically unrealistic, it is necessary that the term inside the braces on the right hand side of (70) be zero; that is,

$$\frac{\frac{\partial u/u_e}{\partial \zeta}_w}{\delta^2 u'_{e,s} Re_L} - (3G_7 G_8 + G_7) + \psi^2 G_{11} - 2\psi \frac{\omega}{u'_{e,s}} G_5 + \frac{\omega^2}{u'^2_{e,s}} G_6 = 0 \quad (71)$$

When the relations (66) and (67) are used in (61) and the same procedure used that was used to obtain (71) the result is

$$\frac{\frac{\partial v/u_e}{\partial \zeta}_w}{\delta^2 u'_{e,s} Re_L} + 4\psi G_{13} + 4 \frac{\omega}{u'_{e,s}} G_9 = 0 \quad (72)$$

The ratio $\frac{\partial v/u_e}{\partial \zeta}_w$ can be replaced by an expression involving $\frac{\partial u/u_e}{\partial \zeta}_w$ by using (56) with "a" = 0. That is

$$v = u_e \psi (\zeta^n - \zeta^{n+1}) + \omega r_0 (1 - \zeta^{n+1}) \quad (73)$$

Then

$$\frac{\partial v/u_e}{\partial \zeta} = \psi [n\zeta^{n-1} - (n+1)\zeta^n] - \omega r_0 (n+1)\zeta^n \quad (74)$$

Also, from (55)

$$\frac{\partial u/u_e}{\partial \zeta} = n\zeta^{n-1} \quad (75)$$

Therefore from (74) and (75)

$$\left(\frac{\partial v/u_e}{\partial \zeta}\right)_w = \psi \left(\frac{\partial u/u_e}{\partial \zeta}\right)_w \quad (76)$$

Then, with (76), Equation (72) becomes

$$\frac{\psi \left(\frac{\partial u/u_e}{\partial \zeta}\right)_w}{\delta^2 u'_{e,s} Re_L} + 4\psi G_{13} + \frac{4\omega}{u'_{e,s}} G_9 = 0$$

$$\psi_s = \frac{\frac{-4\omega G_9}{u'_{e,s}}}{\left(\frac{\partial u/u_e}{\partial \zeta}\right)_w \frac{1}{\delta^2 u'_{e,s} Re_L} + 4G_{13}} \quad (77)$$

when ψ from (77) is substituted into (71) the result can be written as

$$\begin{aligned} & \frac{1}{\delta^2 u'_{e,s} Re_L} \left(\frac{\partial u/u_e}{\partial \zeta} \right)_w - (3G_7 G_8 + G_7) \\ & + \left(\frac{\omega}{u'_{e,s}} \right)^2 \left\{ \frac{16G_{11} G_9^2}{\left[\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w \left(\frac{1}{\delta^2 u'_{e,s} Re_L} \right) + 4G_{13} \right]^2} \right. \\ & \left. + \frac{8G_9 G_5}{\left[\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w \left(\frac{1}{\delta^2 u'_{e,s} Re_L} \right) + 4G_{13} \right]} + G_6 \right\} = 0 \end{aligned} \quad (78)$$

Equation (78) is solved for $\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w \frac{1}{\delta^2 u'_{e,s} Re_L}$ by iteration. When a value is taken for $\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w$, δ is known. In the present investigation $\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w$ is taken as unity which is consistent with laminar flow at the stagnation point and a "power" profile. When $\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w \frac{1}{\delta^2 u'_{e,s} Re_L}$ is known, a substitution into (77) gives ψ .

Thus δ_s and ψ_s can be found for $\alpha = 0$, both for spin and no spin.

The value of δ_s for $\alpha \neq 0$, $\omega = 0$ is found by similar procedure. For $\omega = 0$, the parameters B and C are zero and $A = q_e$. Near the stagnation point the flow outside the boundary layer can be expressed as (p. 462 Reference 10)

$$u_e = c_1 s_x \quad (79)$$

$$v_e = c_2 s_\theta \quad (80)$$

where s_x and s_θ are distances measured from the stagnation point along $\theta = \text{constant}$ and $x = \text{constant}$, respectively. The equation for a streamline is

$$\frac{ds_\theta}{ds_x} = \frac{v_e}{u_e} \quad (81)$$

or with (79) and (80),

$$\frac{ds_\theta}{ds_x} = c_3 \frac{s_\theta}{s_x} \quad (82)$$

where $c_3 = c_2/c_1$. Equation (82) results in

$$s_\theta = c_4 s_x^{c_3} \quad (83)$$

or

$$\frac{ds_\theta}{ds_x} = c_4 c_3 s_x^{c_3-1} \quad (84)$$

At a stagnation point that is off the axis of revolution of an elongated body of revolution, the radius of curvature of the surface in a plane $\theta = \text{constant}$ is greater than the radius of curvature in a plane $x = c$. Consequently $c_2 > c_1$ and so $c_3 > 1$. Therefore all the

streamlines except the one for $c_4 = \infty$ have $\frac{ds_\theta}{ds_x} = 0$ at the stagnation

point (see Figure 4). Thus for all the streamlines except the one for $c_4 = \infty$, the angle α is zero at the stagnation point. The stagnation point is a nodal point of attachment (p 76 Reference 10).

The values for the J's at the stagnation point are obtained by putting $\omega = 0$ in (66). Moreover, going to the stagnation point but not putting $u_e = 0$ yet, it can be shown that

$$\frac{\partial J_1}{\partial x} = 0$$

$$\frac{\partial J_6}{\partial \theta} = -u_e^2 G_{11} \frac{\partial a}{\partial \theta}$$

$$\frac{\partial J_2}{\partial x} = -u_e^2 G_1 \frac{\partial a}{\partial x}$$

$$\frac{\partial J_7}{\partial \theta} = 2u_e \frac{\partial u_e}{\partial \theta} G_{14}$$

$$\frac{\partial J_3}{\partial x} = 2u_e \frac{\partial u_e}{\partial x} G_8 G_7$$

$$\frac{\partial J_8}{\partial \theta} = u_e^2 G_{12} \frac{\partial a}{\partial \theta}$$

(85)

$$\frac{\partial J_6}{\partial x} = -u_e^2 G_{11} \frac{\partial a}{\partial x}$$

$$\frac{\partial J_9}{\partial \theta} = 2u_e \frac{\partial u_e}{\partial \theta} G_{11}$$

$$\frac{\partial J_{14}}{\partial x} = -2u_e \frac{\partial u_e}{\partial x} G_{13}$$

$$\frac{\partial J_{12}}{\partial \theta} = -u_e^2 (2G_{13} - G_2) \frac{\partial a}{\partial \theta}$$

$$\frac{\partial J_{15}}{\partial x} = u_e^2 (2G_9 - G_8) \frac{\partial a}{\partial x}$$

$$\frac{\partial J_{13}}{\partial \theta} = 0$$

These relations together with

$$\tau_{w_0} = \psi \tau_{w_x}$$

are used (61) and the resulting equation divided through by u_e^2 . The result is

$$\begin{aligned}
 & -r_o \cos \phi \frac{\partial \delta}{\partial x} G_{13} \psi - r_o \cos \phi \frac{\partial \psi}{\partial x} \delta G_{13} + \frac{\partial \delta}{\partial \theta} \psi^2 G_{11} + 2\psi \frac{\partial \psi}{\partial \theta} \delta G_{11} \\
 & = r_o \frac{\tau_{wx}}{u_e^2} \psi + \delta \left[\frac{2}{u_e} \frac{\partial u_e}{\partial x} G_{13} \psi r_o \cos \phi + 2\psi G_{13} \frac{\partial r_o}{\partial x} \cos \phi + \psi^2 G_{11} \frac{\partial a}{\partial x} r_o \cos \phi \right. \\
 & + (2G_9 - G_8) \frac{\partial a}{\partial x} r_o \cos \phi + \frac{\partial v_e}{\partial x} \frac{1}{u_e} G_7 r_o \cos \phi - \psi^2 \frac{2}{u_e} \frac{\partial u_e}{\partial \theta} G_{11} + \psi (2G_{13} - G_2) \frac{\partial a}{\partial \theta} \\
 & \left. + \psi \frac{G_2}{u_e} \frac{\partial v_e}{\partial \theta} \right]
 \end{aligned} \tag{86}$$

Now use

$$\frac{\tau_{wx}}{u_e^2} = \frac{\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w}{Re_L \delta u_e} \quad (\text{See Equation (69)}) \tag{87}$$

Also, on the line $\theta = \pi$

$$\frac{\partial a}{\partial x} = 0 \quad (\text{Appendix A}) \tag{88}$$

and at the stagnation point

$$\left(\frac{\partial a}{\partial \theta}\right)_s = -\frac{1}{2} \frac{\frac{\partial^2 u_e}{\partial \theta^2}}{\left(\frac{\partial v_e}{\partial \theta}\right)_s} \quad (\text{Appendix A}) \quad (89)$$

Because neither $\left(\frac{\partial^2 u_e}{\partial \theta^2}\right)_s$ nor $\left(\frac{\partial v_e}{\partial \theta}\right)_s$ is zero (Appendix A) the term $\left(\frac{\partial a}{\partial \theta}\right)_s$ is neither zero nor infinite. When (87), (88), (89), and $\frac{1}{u_e} \frac{\partial v_e}{\partial x} = 0$ (Appendix A) are used, the right hand side of (86) becomes

$$\psi \left\{ r_o \frac{\left(\frac{\partial u/u_e}{\partial \zeta}\right)_w}{\text{Re}_L \delta u_e} + \delta \left[\frac{2}{u_e} \frac{\partial u_e}{\partial x} G_{13} r_o \cos \phi + 2G_{13} \frac{\partial r_o}{\partial x} \cos \phi - \frac{2\psi}{u_e} \frac{\partial u_e}{\partial \theta} G_{11} \right. \right. \\ \left. \left. + (2G_{13} - G_2) \frac{\partial a}{\partial \theta} + \frac{G_2}{u_e} \frac{\partial v_e}{\partial \theta} \right] \right\} \quad (90)$$

To prevent (90) from becoming infinite as u_e approaches zero at the stagnation point it is necessary that $\psi = 0$ at the stagnation point. Therefore the value of ψ at a stagnation point for $\alpha \neq 0$, $\omega = 0$ is zero.

When $\psi = 0$ and (66) with $\omega = 0$ and (85) are used in (60) the result is

$$r_o \cos \phi \frac{\partial \delta}{\partial x} u_e^2 G_7 G_8 + \frac{\partial \psi}{\partial \theta} \delta u_e^2 G_{14} = r_o \tau_{w_x} \delta \left(2u_e \frac{\partial u_e}{\partial x} r_o \cos \phi G_7 G_8 \right. \\ \left. + u_e^2 G_7 G_8 \cos \phi \frac{\partial r_o}{\partial x} + r_o \cos \phi u_e \frac{\partial u_e}{\partial x} G_7 + u_e^2 G_{12} \frac{\partial a}{\partial \theta} \right) \quad (91)$$

Equation (91) is divided through by u_e^2 . The term τ_{wx} becomes $\frac{\tau_{wx}}{u_e^2}$

which is written as $\frac{1}{u_e \delta Re_L} \left(\frac{\partial u/u_e}{\partial \zeta} \right)_w$. The relation $u_e = u'_{e,s} s_x$ where s_x is measured from the stagnation point is also used, as is the relation $G_{12} = G_7 G_8$. Then (91) becomes

$$r_o \frac{\partial \delta}{\partial s_x} G_{12} + \frac{\partial \psi}{\partial \theta} \delta G_{14} = \frac{r_o}{u'_{e,s} s_x \delta Re_L} \left(\frac{\partial u/u_e}{\partial \zeta} \right)_w - \delta \left[\frac{r_o}{s_x} (2G_7 G_8 + G_7) + G_7 G_8 \left(\frac{\partial r_o}{\partial s_x} + \frac{\partial a}{\partial \theta} \right) \right] \quad (92)$$

In order to prevent the right hand side of (92) from becoming infinite as s_x becomes zero it is necessary that

$$\frac{1}{u'_{e,s} \delta Re_L} \left(\frac{\partial u/u_e}{\partial \zeta} \right)_w - \delta (2G_7 G_8 + G_7) = 0 \quad (93)$$

or

$$\delta_s = \left[\frac{\left(\frac{\partial u/u_e}{\partial \zeta} \right)_w}{u'_{e,s} Re_L (2G_7 G_8 + G_7)} \right]^{\frac{1}{2}} \quad (94)$$

Thus, for $\alpha \neq 0$, $\omega = 0$ the value of δ at the stagnation point is given by (94). The starting value of δ can now be obtained from (65), (78), and (94).

For $\alpha = 0$ the starting value of ψ is gotten from (77). For $\alpha \neq 0$, $\omega = 0$ the starting value of ψ is zero. For $\alpha \neq 0$, $\omega \neq 0$ the value of ψ at the stagnation point is obtained by use of (39). Thus

for all streamlines except the one for $c_4 = \infty$ (See (83) and Figure 4) v_q lies in the plane $x = \text{constant}$ through the stagnation point because $a = 0$ there. Therefore at the surface $v_q = \omega r_0$ and $(\frac{\partial v_q}{\partial y}) < 0$ (see (46),

(47)). Also $u_q = 0$ for all y at the stagnation point; therefore $\frac{\partial u_q}{\partial y} = 0$ for all y . Thus it follows from (39) that $\psi = -\infty$ for all streamlines at the stagnation point except the one for $c_4 = \infty$. For $c_4 = \infty$, $a = \pi/2$ and similar reasoning results in $\psi = 0$. Thus ψ is double valued at the stagnation point. The integrations are made for $c_4 = \infty$ so that $\psi = -\infty$ is the starting value.

Friction Coefficient

The component of the friction coefficient in the direction of q_e , the velocity at the outer edge of the boundary layer, is approximated by a formula based on δ and on the magnitude of the component in the direction of q_e of the relative velocity of q_e with respect to the surface. The formula is

$$\frac{\bar{\tau}_{wq_e}}{\bar{\rho}[\bar{q}_e - \bar{u}_{q,w}]^2} = \frac{k}{\left[\frac{\delta |\bar{q}_e - \bar{u}_{q,w}|}{\bar{v}} \right]^m} \quad (95)$$

or

$$\frac{\bar{\tau}_{wq_e}}{\bar{\rho} \bar{v}_\infty^2} = \tau_{wq_e} = \frac{k}{Re_L^m} \frac{|A|^{2-m}}{\delta^m} \quad (96)$$

Equation (96) is consistent with the expression (32) for the velocity profile in the direction of q_e . The component τ_{wx} along a meridian is given by (see Figure 5)

$$\tau_{wx} = \tau_{wq_e} (\cos a - \psi \sin a) \quad (97)$$

and the component in a plane $x = \text{constant}$ by

$$\tau_{w\theta} = \tau_{wq_e} (\sin a + \psi \cos a) \quad (98)$$

Integration of Equation for Displacement Surface

The quantity, Q , in Equation (26) is the right hand side of Equation (24). The integrals $\int_0^\delta (u_e - u) dy$ and $\int_0^\delta (v_e - v) dy$ are given in Table II. The Equation (26) becomes

$$\frac{d\delta^*}{dx} = - \frac{Q}{r_o \cos \phi u_e}$$

or

$$\begin{aligned} \cos \phi \frac{d\delta^*}{dx} = & -\delta^* \left(\frac{\cos \phi}{u_e} \frac{\partial u_e}{\partial x} + \frac{\cos \phi}{r_o} \frac{\partial r_o}{\partial x} + \frac{1}{r_o u_e} \frac{\partial v_e}{\partial \theta} \right) \\ & + \frac{\delta}{u_e} \cos \phi \left(\psi \frac{\partial J_4}{\partial x} + J_4 \frac{\partial \psi}{\partial x} + \frac{\partial J_5}{\partial x} \right) + \frac{\cos \phi}{u_e} \left(\frac{\partial \delta}{\partial x} + \frac{\delta}{r_o} \frac{\partial r_o}{\partial x} \right) (\psi J_4 + J_5) \\ & + \frac{\delta}{r_o u_e} \left(J_{18} \frac{\partial \psi}{\partial \theta} + \psi \frac{\partial J_{18}}{\partial \theta} + \frac{\partial J_{19}}{\partial \theta} \right) + \frac{1}{r_o u_e} \frac{\partial \delta}{\partial \theta} (J_{18} \psi + J_{19}) \end{aligned} \quad (99)$$

The quantities δ , ψ , and their derivatives are obtained by use of (60), (61), (62) and (63). The derivatives of the J 's are obtained

by calculating the J 's at $x+\Delta x$ and $\theta+\Delta\theta$ and using $\frac{\partial J}{\partial x} \cong \frac{\Delta J}{\Delta x}$ and

$\frac{\partial J}{\partial \theta} \cong \frac{\Delta J}{\Delta \theta}$. The velocity derivatives and $\frac{\partial r_o}{\partial x}$ are obtained from the

velocity distribution and shape of the body. The initial value of

δ^* is taken as $\frac{\delta_s}{2}$ which is valid for a power profile with $n=1$, the

approximation to a laminar profile at the stagnation point. In (99), the derivative $\frac{d\delta^*}{dx}$ is along a streamline. Thus (27), (60), (61), and (99) are integrated simultaneously step-by-step. The result is δ , ψ , δ^* , and the streamline path $\theta(x)$. Different streamlines are obtained by beginning the integrations at points on a circle of small radius in the tangent plane at the stagnation point. Through each point on the circle goes only one streamline; the entire streamline is thus fixed by choosing the point on the tangent circle. The relation for the initial value of θ in the integration is

$$\theta = \pi + \Delta\theta = \pi - \frac{I}{r} \sin \eta \quad (100)$$

and the relation for the initial value of x is

$$x = x_s + \Delta x = x_s + I \cos\phi \cos \eta \quad (101)$$

The angle η is zero along the meridian $\theta=\pi$ in the direction of positive x . The distance I is the radius of the starting circle.

Derivation of Equations for Magnus Force and Moment

After the displacement surface thickness δ^* has been computed it is added to the body. To calculate the force and moment on the resulting body, slender-body theory (References 5 and 6) can be used. In slender-body theory the force acting on the portion of the body between the nose and a section (C-C) perpendicular to the ξ axis (Figure 6) is given by

$$\frac{\bar{F}}{\frac{\bar{V}_\infty}{2} L^2} = -2i \int_C \phi_0 dz_0 \quad (\text{page 50, Reference 6}) \quad (102)$$

Here ϕ_0 is the perturbation potential in the z_0 plane and is a solution of Laplace's equation.

Equation (102) can be placed in a form more convenient for use in the present analysis by integrating by parts. Then (102) becomes

$$\frac{\bar{F}}{\frac{\bar{\rho} \bar{V}_\infty^2}{2} \bar{L}^2} = -2i [z_{01} (\Delta \phi_0) - \int_c z_0 \frac{d\phi_0}{ds_\xi} ds_\xi] \quad (103)$$

where $\Delta \phi_0$ is the change in ϕ_0 in going around the section of the body, starting the ending at the point z_{01} . Because $\Delta \phi_0$ is zero, (See 9.3.15 of Reference 5), Equation (103) becomes

$$\frac{\bar{F}}{\frac{\bar{\rho} \bar{V}_\infty^2}{2} \bar{L}^2} = 2i \int_c z_0 V'_s ds_\xi \quad (104)$$

In order to use (104), V'_s must be known along the contour c .

To find V'_s along c the velocity vector \vec{Q} is written as

$$\vec{Q} = v_\infty (1 + \frac{\partial \phi}{\partial \xi}) \hat{e}_1 + v_\infty \frac{\partial \phi}{\partial \eta} \hat{\eta} \quad (105)$$

where ϕ is the perturbation potential, \hat{e}_1 is a unit vector along the ξ axis which is along \vec{V}_∞ , and the unit vector $\hat{\eta}$ is normal to the body cross section in a plane $\xi=c$ (Figure 6). The unit normal \hat{n} to the surface of the body is

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} \quad (106)$$

where $T(\xi, h, \beta) = c$ is the equation of the body plus the displacement surface. Because there is no flow through the surface of the body

$$\vec{Q} \cdot \hat{n} = 0 \quad (107)$$

which, with (105) and (106), becomes

$$(1 + \frac{\partial \Phi}{\partial \xi}) \hat{e}_1 \cdot \nabla T + \frac{\partial \Phi}{\partial v} \hat{v} \cdot \nabla T = 0 \quad (108)$$

To obtain an expression for \hat{v} (Figure 6), the unit vector along s_ξ is written as

$$\hat{s} = \hat{h} \frac{dh_o}{ds_\xi} + \hat{\beta} h \frac{d\beta}{ods_\xi} \quad (109)$$

Then

$$\hat{v} = \hat{e}_1 \cdot \hat{s} = -\hat{\beta} \frac{dh_o}{ds_\xi} + \hat{h} h \frac{d\beta}{ods_\xi} \quad (110)$$

The gradient ∇T is written as

$$\nabla T = \hat{e}_1 T_\xi + \hat{h} T_h + \frac{\hat{\beta}}{h} T_\beta \quad (111)$$

Then (108) becomes

$$(1 + \frac{\partial \phi}{\partial \xi}) T_{\xi} + \frac{\partial \phi}{\partial v} (T_h h_o \frac{d\beta}{ds_{\xi}} - \frac{T_{\beta}}{h_o} \frac{dh_o}{ds_{\xi}}) = 0 \quad (112)$$

According to slender-body theory the body has a small slope in a plane $\beta=c$, therefore $T_{\xi} \ll 1$. Also the disturbance velocity $\frac{\partial \phi}{\partial \xi}$ is small. Therefore the product $\frac{\partial \phi}{\partial \xi} T_{\xi}$ in (112) is neglected with the result that

$$\frac{\partial \phi}{\partial v} = - \frac{T_{\xi}}{T_h h_o \frac{d\beta}{ds_{\xi}} - \frac{T_{\beta}}{h_o} \frac{dh_o}{ds_{\xi}}} \quad (113)$$

But

$$ds_{\xi} = \sqrt{1 + (\frac{1}{h_o} \frac{dh_o}{d\beta})^2} h_o d\beta \quad (\xi=c) \quad (114)$$

Also

$$T(\xi, h_o, \beta) = h_o - h_o(\xi, \beta) = c$$

Then

$$T_{\xi} = -\left(\frac{\partial h_0}{\partial \xi}\right)_{\beta}$$

and

$$T_{\beta} = -\left(\frac{\partial h_0}{\partial \beta}\right)_{\xi}$$

and

$$T_h = 1$$

Then (113) becomes

$$\frac{\partial \phi_0}{\partial v} = \frac{\left(\frac{\partial h_0}{\partial \xi}\right)_{\beta}}{\sqrt{1 + \frac{1}{h_0^2} \left(\frac{\partial h_0}{\partial \beta}\right)_{\xi}^2}} \quad (115)$$

where ϕ in (113) has been replaced by ϕ_0 . According to the approximations of slender-body theory, it is permissible to replace the three-dimensional disturbance potential ϕ in (113) by ϕ_0 the two-dimensional disturbance potential that is a solution of Laplace's equation in a plane $\xi=c$.

When the displacement surface has been added to the body, Equation (115) is used to obtain $\frac{\partial \phi_0}{\partial v}$, which equals V_n' the non-dimensional velocity normal to a section $\xi=c$. The expressions for

$(\frac{\partial h_0}{\partial \xi})_\beta$, h_0 , and $(\frac{\partial h_0}{\partial \beta})_\xi$ are derived in Appendix B. Equation (115) gives the non-dimensional velocity normal to a body section $\xi=c$. In the calculation of $(\frac{\partial h_0}{\partial \xi})_\beta$ and $(\frac{\partial h_0}{\partial \beta})_\xi$ the quantity ξ occurs (see B-21 and B-22). The boundary layer calculations provide δ^* and other quantities as functions of x . Consequently ξ is found from a specified value of x . The value of ξ for a fixed value of x , however, depends on θ (See B-39). Consequently when x is fixed the value of ξ varies slightly with θ . The normal velocity $\frac{\partial \phi_0}{\partial v}$ therefore is obtained on a curve that does not lie exactly in a plane $\xi=c$. By putting $R=r_0+\delta^*$ in (B-39) and noting that δ^* , $\sin \alpha$, and r_0 are all small of $o(\alpha)$, it follows that the products $r_0 \sin \alpha$ and $\delta^* \sin \alpha$ are of the $o(\alpha^2)$. Therefore, the departure from the plane

$$\xi_c = (d+x_0) \cos \alpha$$

of the curve on which $\frac{\partial \phi_0}{\partial v}$ is calculated is of $o(\alpha^2)$, a small quantity when α is small.

If the value of ξ were fixed at the value given by

$$\xi_c = (d+x_0) \cos \alpha$$

the values of x such that $\xi=\xi_c$ could be found by putting $\xi=\xi_c=(d+x_0) \cos \alpha$ in (B-39) and solving for x to obtain

$$x = x_0 - R(x, \theta) \cos \theta \tan \alpha \quad (116)$$

The values of $\frac{\partial \delta^*}{\partial x}$ and $\frac{\partial \delta^*}{\partial \theta}$ for $x(\theta)$ would then be found by interpolation from the computed values of $\frac{\partial \delta^*}{\partial x}$ and $\frac{\partial \delta^*}{\partial \theta}$ at two values of x , one larger and one smaller than the largest and smallest value of x given by (116). This procedure was not carried out in the example calculation for the half-ellipsoid because the additional complication it would have introduced into the calculations was judged not to be justifiable. In the calculation of $(\frac{\partial h_0}{\partial \beta})_\xi$, $(\frac{\partial h_0}{\partial \xi})_\beta$ etc. (Appendix B),

the exact expressions involve hardly any more computation than expressions that would be obtained by neglecting terms of $o(\alpha^2)$. Consequently the exact expressions were used.

The next step in the process of finding V'_S for use in (104) is to move the origin of coordinates in a plane $\xi=c$ back to the axis of revolution, that is, to the x axis (see Figure 7). The equations for doing this are

$$r' = \sqrt{h_0^2 + 2h_0(1+x)\sin\alpha\cos\beta + (d+x)^2\sin^2\alpha} \quad (117)$$

and

$$\cos\gamma = \frac{-h_0\sin\beta}{r'} \quad (118)$$

At this stage in the calculation the vertical velocity V'_n along the distorted circle with its center on the x axis and lying in a $\xi = \text{constant}$ plane is known. To find V'_S from V'_n conformal transformation theory as applied in Reference 11, 12 and 13 is used. The method of Reference 11 and 12 is used to transform the distorted circle in the r', γ plane into a circle in the λ, σ plane (Figure 7). Because

$$v'_n = \frac{\partial\phi_0}{\partial v} \approx \frac{\Delta\phi_0}{\Delta v} \quad (119)$$

and

$$v_n = \frac{\partial\phi_0}{\partial\lambda} \approx \frac{\Delta\phi_0}{\Delta\lambda} \quad (120)$$

and because the potential ϕ_0 at corresponding points in the Z' and Z planes (Figure 7) is unchanged by a conformal transformation of

the Z' plane into the Z plane (p 163 of Reference 14), it follows from (119) and (120) that

$$\frac{v_n}{v'_n} = \frac{\Delta v}{\Delta \lambda} \quad (121)$$

But $\Delta v = |dz'|$ and $\Delta \lambda = |dz|$; therefore (121) is

$$\frac{v_n}{v'_n} = \left| \frac{dz'}{dz} \right| \quad (122)$$

The ratio $\left| \frac{dz'}{dz} \right|$ for points on the cross sections in the Z and Z' planes is given by (Equation (37) of Reference 12), namely

$$\left| \frac{dz'}{dz} \right| = e^{\Omega - \Omega_0} \frac{\sqrt{1 + \left(\frac{d\Omega}{d\gamma} \right)^2}}{1 + \frac{d\epsilon}{d\gamma}} \quad (123)$$

where for points on the cross sections in the Z' and Z planes,

$$z' = r' e^{i\gamma} = r_0 e^{\Omega + i\gamma} \quad (124)$$

and

$$z = \lambda e^{i\sigma} = r_0 e^{\Omega_0 + i\sigma} \quad (125)$$

and

$$\varepsilon = \sigma - \gamma \quad (126)$$

The constant Ω_0 is calculated by

(Page 8 of Reference 11 and Equation C of Reference 12)

$$\Omega_0 = \frac{1}{2\pi} \int_0^{2\pi} \Omega(\gamma) d\gamma \quad (127)$$

The quantity ε is calculated from

(Equation 13 Reference 12)

$$\varepsilon(\sigma') = -\frac{1}{2\pi} \int_0^{2\pi} \Omega(\sigma) \cot \frac{\sigma - \sigma'}{2} d\sigma \quad (128)$$

Because δ^* is a small fraction of r_0 , the cross section in the plane $\xi = c$ is almost a circle. Therefore the quantity σ in (128) can be replaced by γ (see Reference 11) so that the equation used to calculate ε is

$$\varepsilon(\gamma') = -\frac{1}{2\pi} \int_0^{2\pi} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma \quad (129)$$

For more distorted cross sections iteration is used (see Reference 12).

Once V_n is known on the circle in the λ, σ plane, the velocity V_s can be calculated from an equation given on page 5 of Reference 13; the equation is

$$V_s(\sigma') = -\frac{1}{2\pi} \int_0^{2\pi} V_n(\sigma) \cot \frac{\sigma-\sigma'}{2} d\sigma \quad (130)$$

When $V_s(\sigma)$ is known from (130), $V_s'(r', \gamma)$ is found from

$$V_s'(r', \gamma) = V_s(\sigma) \left| \frac{dz}{dz'} \right| \quad (131)$$

where $\left| \frac{dz}{dz'} \right|$ is the reciprocal of $\left| \frac{dz'}{dz} \right|$ given by (123). Then because $V_s'(r', \gamma) = V_s'(h_0, \beta)$ at corresponding points, the integral in (104) can be evaluated.

Actually it is more convenient to express the integral in (104) in terms of quantities in the (λ, σ) plane. From (131)

$$V_s'(r', \gamma) = V_s(\sigma) \frac{ds}{ds_\xi} \quad (132)$$

But $V_s'(h_0, \beta) = V_s'(r', \gamma)$ and $ds_\xi = ds_\xi'$ because the z_0 and z' planes are identical except for a shift in origin (see Figure 7). Therefore by using (132) the term $V_s' ds_\xi$ in (104) can be written as

$$V_s' ds_\xi = V_s(\sigma) ds \quad (133)$$

Also, from Figures 7 and 8

$$z_0 = z' - (d+x) \sin \alpha \quad (134)$$

or with (124)

$$z_0 = r_0 e^{\Omega + i\gamma} - (d+x) \sin \alpha \quad (135)$$

Then (104) becomes

$$\frac{\frac{\bar{F}}{2}}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \frac{\bar{L}^2}{2}} = 2i \int_c [r_0 e^{\Omega + i\gamma} - (d+x) \sin \alpha] V_s ds \quad (136)$$

From (133)

$$V_s ds = V'_s ds_\xi$$

But

$$\int_c V'_s ds_\xi = \Delta \Phi_0 = 0 \quad (\text{see Equation (103)})$$

Therefore (136) is

$$\frac{\frac{\bar{F}}{2}}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \frac{\bar{L}^2}{2}} = 2i r_0 \int_c e^{\Omega} (\cos \gamma + i \sin \gamma) V_s ds \quad (137)$$

or with $r_o e^{\Omega} = r'(\gamma)$ and $ds = \lambda d\sigma$

$$\frac{\bar{F}_z}{\frac{\bar{\rho}_{\infty} \bar{V}_{\infty}^2}{2} L} = 2\lambda \int_c r' v_s \cos \gamma d\sigma \quad (138)$$

and

$$\frac{\bar{F}_y}{\frac{\bar{\rho}_{\infty} \bar{V}_{\infty}^2}{2} L} = -2\lambda \int_0^{2\pi} r' v_s \sin \gamma d\sigma \quad (139)$$

where

$$\frac{r'}{r_o} = e^{\Omega} \quad \text{and} \quad \frac{\lambda}{r_o} = e^{\Omega_o} \quad (140)$$

and Ω and γ are known functions of σ because $\Omega = \Omega(\gamma)$ and $\gamma = \sigma - \epsilon$.

The force \bar{F}_y is the Magnus force for the section of the body between the nose and the section x . The force \bar{F}_z lies in the plane through the free stream velocity vector and the axis of revolution of the body.

The Magnus moment about the nose of the body is given by the expression, with \bar{L} taken as the body length,

$$\frac{\bar{M}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^3}{2}} = \int_{-d}^0 (d+x) \frac{d\left(\frac{\bar{F}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^2}{2}}\right)}{dx} dx \quad (141)$$

where $\frac{\bar{F}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^2}{2}}$ is given by (139). By integration by parts

Equation (141) can be placed in the more convenient form

$$\frac{\bar{M}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^3}{2}} = d \left(\frac{\bar{F}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^2}{2}} \right)_{x=0} - \int_{-d}^0 \frac{\bar{F}_Y}{\frac{\bar{\rho} \bar{V}_\infty^2 \bar{L}^2}{2}} dx \quad (142)$$

Once the boundary layer displacement surface surrounding the body has been calculated, the Magnus force and moment can be calculated by a basically more exact method than slender-body theory, namely potential flow theory as applied in Reference 7. Although this method was not used in the present investigation because of a lack of funds, its use is discussed in Appendix C.

CALCULATION METHOD APPLIED TO FINENESS RATIO 5 HALF-ELLIPSOID

To begin the numerical integration the value of I in (100) and (101) is taken as 10^{-5} for all η except $\pm 95^\circ$. For $\eta = \pm 95^\circ$ I is taken as 10^{-3} in order to increase the spacing between the $\eta = \pm 90^\circ$ and $\eta = \pm 95^\circ$ streamlines (Figure 9).

The increments in θ and δ^* are calculated from (27) and (99) by use of the relations

$$\Delta\theta = \left(\frac{d\theta}{dx}\right)\Delta x$$

and

$$\Delta\delta^* = \frac{d\delta^*}{dx} \Delta x$$

The values of $\Delta\delta$ and $\Delta\psi$ are obtained from (62) and (63). The value of Δx used to calculate $\Delta\theta, \Delta\delta^*, \Delta\delta$ and $\Delta\psi$ is determined by the rate of change of $\frac{d\theta}{dx}$ between two succeeding values of x . The criterion is:

$$(a) \text{ if } .0002 < \left| 1 - \frac{\left(\frac{d\theta}{dx}\right)_{x-\Delta x}}{\left(\frac{d\theta}{dx}\right)_x} \right| < .002, \text{ then}$$

Δx is left unchanged

$$(b) \text{ if } \left| 1 - \frac{\left(\frac{d\theta}{dx}\right)_{x-\Delta x}}{\left(\frac{d\theta}{dx}\right)_x} \right| < .0002, \text{ then } \Delta x \text{ is doubled for the}$$

next integration step

$$(c) \text{ if } \left| 1 - \frac{\left(\frac{d\theta}{dx}\right)_{x-\Delta x}}{\left(\frac{d\theta}{dx}\right)_x} \right| > .002, \text{ then } \Delta x \text{ is halved for the}$$

next integration step

In no case is Δx allowed to exceed 10^{-4} .

To calculate the derivatives of the J 's (Equation 60, 61) $\Delta\theta$ and Δx are both taken as 10^{-4} and the derivatives are calculated by

$$\frac{\partial J_i}{\partial x} = \frac{J_{i,(x+\Delta x)} - J_{i,x}}{\Delta x}$$

$$\frac{\partial J_i}{\partial \theta} = \frac{J_{i,(\theta+\Delta\theta)} - J_{i,\theta}}{\Delta\theta}$$

When Equations (27), (60), (61) and (99) were integrated, it was found that rapidly diverging oscillations occurred at the start. This caused the computer to stop. Rather than spend time trying to eliminate the divergence, the values of δ , ψ , and δ^* were fixed at their stagnation point values until x became larger than $-.998$. They were then allowed to vary and no oscillations occurred. The stagnation point is at $-.999910$ so that δ , ψ , and δ^* were fixed for the first $.001910$ in x . For the largest value of η , namely $\pm 95^\circ$, a calculation shows that the distance along the streamline is about $.0057$ compared with $.00191$ along x . Thus for the streamlines calculated (Figure 9), δ , ψ , and δ^* are fixed for distances along streamlines not greater than $.0057$ from the stagnation point. Consequently the results for δ^* , and so for the Magnus force on the portion of the body near the nose, may be inaccurate. Effects of changes in δ , ψ , and δ^* near the stagnation point decrease rapidly with increase in x . Thus, a change in δ at the stagnation point by a factor of 5 for $\eta=95^\circ$ makes only a change of 3 in the fourth significant figure in δ^* for $x = -.600$.

To evaluate the integral in (129) and (130) the method of Reference 11 is followed. Equation (129) is integrated by first noting that the integrand becomes infinite at $\gamma' = \gamma$. Consequently (129) is written as

$$\varepsilon(\gamma') = -\frac{1}{2\pi} \left[\int_0^{\gamma_1} \Omega(\gamma) \cot \frac{\gamma-\gamma'}{2} d\gamma + \int_{\gamma_2}^{2\pi} \Omega(\gamma) \frac{\gamma-\gamma'}{2} d\gamma + \right]$$

$$+ \lim_{\eta_0 \rightarrow 0} \left\{ \int_{\gamma_1}^{\gamma'_1 - \eta_0} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma + \int_{\gamma'_1 + \eta_0}^{\gamma_2} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma \right\} \quad (143)$$

Then near $\gamma = \gamma'$, $\Omega(\gamma)$ is written as

$$\Omega(\gamma) = \Omega(\gamma') + (\gamma - \gamma') \frac{d\Omega}{d\gamma}(\gamma') + \dots$$

and substituted into

$$\lim_{\eta_0 \rightarrow 0} \left\{ \int_{\gamma_1}^{\gamma'_1 - \eta_0} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma + \int_{\gamma'_1 + \eta_0}^{\gamma_2} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma \right\}$$

thereby giving $4\Delta \frac{d\Omega}{d\gamma}(\gamma')$, where $\gamma'_1 - \gamma_1 = \Delta = \gamma_2 - \gamma'_1$.

Then equation (143) becomes

$$\begin{aligned} \epsilon(\gamma') = - \frac{1}{2\pi} & \left[\int_0^{\gamma'_1 - \Delta} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma + \int_{\gamma'_1 + \Delta}^{2\pi} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma \right. \\ & \left. + 4\Delta \frac{d\Omega}{d\gamma}(\gamma') \right] \quad (144) \end{aligned}$$

The integrals $\int_0^{\gamma' - \Delta} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma$ and $\int_{\gamma' + \Delta}^{2\pi} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma$ are evaluated

by dividing the interval of integration into equal lengths equal to Δ and taking Ω equal to the average of its values at the beginning and end of each interval of length Δ .

Then

$$\int_{k\Delta}^{(k+1)\Delta} \Omega(\gamma) \cot \frac{\gamma - \gamma'}{2} d\gamma = \Omega_{k+1} \ln \frac{\sin \frac{(k+1)\Delta - \gamma'}{2}}{\sin \frac{k\Delta - \gamma'}{2}}$$

$$\text{where } \Omega_{k+1} = \Omega[(k+1)\Delta] + \Omega[k\Delta] \quad (145)$$

By using (145) in (144) the expression for $\varepsilon(\gamma')$ becomes

$$\varepsilon(n) = -\frac{1}{2\pi} \left[4\Delta \frac{d\Omega}{d\gamma}(n) + \sum_{k=1}^{n-1} \Omega_k \ln \frac{\sin(k-n)\frac{\Delta}{2}}{\sin(k-1-n)\frac{\Delta}{2}} + \sum_{k=n+2}^N \Omega_k \ln \frac{\sin(k-n)\frac{\Delta}{2}}{\sin(k-1-n)\frac{\Delta}{2}} \right] \quad (146)$$

where $N = \frac{2\pi}{\Delta}$. For $n=0$, instead of (146) the result is

$$\varepsilon(o) = -\frac{1}{2\pi} \left[4\Delta \frac{d\Omega}{d\gamma}(o) + \sum_{k=2}^{N-1} \Omega_k \ln \frac{\sin \frac{k\Delta}{2}}{\sin \frac{(k-1)\Delta}{2}} \right] \quad (147)$$

Also $\varepsilon(N) = \varepsilon(o)$. In the calculations, N was taken as 180. A test with $N=360$ showed a noticeable but negligible change in $\varepsilon(\gamma)$.

To evaluate the integrals in (138) and (139) the trapezoidal rule was used with the interval 2π divided into 180 equal parts.

The program for the numerical calculations for use on the CDC 6400 is given in Table IV.

RESULTS FOR FINENESS RATIO 5 HALF-ELLIPSOID

As an example, the Magnus force and moment on a fineness ratio 5 half-ellipsoid spinning at $p=.25$ and at $\alpha=4^\circ$ is calculated. The boundary layer is turbulent and is characterized by $n=1/5$. The Reynolds number based on the length of the half-ellipsoid is 4.8×10^6 .

The inviscid flow velocity distribution over the half-ellipsoid is assumed to be the same as over the forward half of a complete ellipsoid. Because of the wake this assumption is not exact, especially near the base, but is used in order to obtain an analytic expression for the velocity distribution in the example calculation which is of an exploratory nature. The velocity distribution is obtained from Chapter V of Reference 15. The parameter λ (Figure 8) equals unity here. The result is

$$u_e = \left[\frac{1-x^2}{1-x^2(1-t^2)} \right]^{\frac{1}{2}} K_1 \cos \alpha - \frac{x}{[1-x^2(1-t^2)]^{\frac{1}{2}}} t K_2 \cos \theta \sin \alpha \quad (148)$$

and

$$v_e = -K_2 \sin \theta \sin \alpha \quad (149)$$

where

$$K_1 = \frac{2(1-t^2)^{3/2}}{2\sqrt{1-t^2} - t^2 \ln \frac{1 + \sqrt{1-t^2}}{1 - \sqrt{1-t^2}}} \quad (150)$$

and

$$K_2 = \frac{4(1-t^2)^{3/2}}{2\sqrt{1-t^2} (1-2t^2) + t^2 \ln \frac{1 + \sqrt{1-t^2}}{1 - \sqrt{1-t^2}}} \quad (151)$$

The half-ellipsoid for which the calculations are made has $t=.1$ so that $K_1=1.02070$ and $K_2=1.96023$. The shape of the half-ellipsoid is given by

$$r_o = t\sqrt{1-x^2}$$

Calculated streamlines on the half-ellipsoid are shown in Figure 9 for positive values of η . Each streamline shown in Figure 9 has a corresponding streamline which is its reflection in the line $\theta=\pi$ and which has a negative value of η .

The boundary layer displacement thickness δ^* along the streamlines shown in Figure 9 ($\eta>0$) are presented for $\eta>0$ in Figure 10a and for $\eta<0$ in Figure 10b.

In Figure 11a and 11b is given the distribution of the full boundary layer thickness δ along various streamlines.

In Figure 12a and 12b is shown the variation of δ^* with θ at $x=-.5$ and at $x=0$. The variation is slightly unsymmetric about $\theta=\pi$; it is this asymmetry that produces the Magnus force.

In Figure 13, (a)-(i) is shown the variation of ψ with x for all the streamlines calculated. The calculations were begun with $\psi=-1$ at the stagnation point instead of $\psi=-\infty$. A few trials showed that in the present example, $\alpha=4^\circ$ and $p=.25$, when the initial value of ψ was less than -5 or so, the integration of the differential equations for δ^* , δ , and ψ could not be made because of divergent oscillations. The behavior of ψ near the stagnation point shown in Figure 13 indicates that the effect of the choice of the initial value of ψ soon disappears. Moreover, two calculations for $\alpha=6^\circ$, $p=1$, $\eta=-10$, one with $\psi_s=-5$ and the other with $\psi_s=-10$ showed that when x had increased to $-.985555$ from the stagnation point value $-.999796$, the values of ψ for the two calculations differed by only 1 in the third significant figure. The difference then decreased as x increased.

In Figure 14 is shown the normal force coefficient C_N based on the local cross section area for the portion of the body between

station x and the nose. The coefficient C_N is obtained from $\frac{\bar{F}_Z}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \bar{L}^2}$ (Equation (137)) by the relation

$$C_N = \frac{\bar{F}_Z}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \bar{L}^2} \left(\frac{\bar{L}^2}{\pi \bar{r}_o^2} \right) \quad (152)$$

In Figure 15 is shown the corresponding Magnus force coefficient C_y

obtained from $\frac{\bar{F}_Y}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \bar{L}^2}$ (Equation (138)) by

$$C_y = \frac{\bar{F}_Y}{\frac{\bar{\rho}_\infty \bar{V}_\infty^2}{2} \bar{L}^2} \left(\frac{\bar{L}^2}{\pi \bar{r}_o^2} \right) \quad (153)$$

Although the sign of C_y computed in the present method is positive along the +Y axis in Figure 6 for ω as shown in Figure 1, it is shown negative in Figure 15 in order to adhere to the sign convention given in Reference 16.

The curves for C_N and C_y are shown dashed for $x < -.6$ because from calculations for $\eta = \pm 95^\circ$, and a calculation for $\eta = +95^\circ$ with δ_s equal to 5 times the correct value of δ_s , it was found that the change in δ^* for $x < -.600$ caused by using $5\delta_s$ instead of δ_s in the calculation for $\eta = +95^\circ$ was greater than 10 percent of the difference between δ^* for $\eta = 95^\circ$ and δ^* for $\eta = -95^\circ$. Thus the Magnus force for $x < -.600$ may be affected by more than 10 percent by inaccuracies in the initial values of δ and ψ and by the method of numerical integration; see section on "Numerical Calculations".

The results in Figures (14) and (15) indicate that because of the boundary layer, C_N decreases and C_y increases as the fineness ratio increases. The decrease in C_N is almost linear with x for larger x but C_y increases more rapidly than in a linear manner all the way to $x=0$, the body base.

The value of the moment coefficient of the entire body defined by (Reference 16),

$$C_n = \frac{\bar{M}_y}{\frac{\rho}{2} V_\infty^2 L^3} \left(\frac{1}{2\pi} \right) \left(\frac{\bar{L}}{r_M} \right)^3 \quad (154)$$

is found to be 1.058×10^{-3} . (See Equation (142))

COMPARISON WITH EXPERIMENT

There do not seem to be any experimental data for a spinning half-ellipsoid of fineness ratio 5. Consequently, strictly speaking there can be no comparison of the calculated results with experiment. There are, however, some data for pointed shapes of near 5 fineness ratio in subsonic flow. One set of data is that of Reference 16. Tests were made of a model with a 2 cal. secant-ogive nose and a 3 cal. cylinder after-body. For $\alpha = 4^\circ$, $M = .2$, and $Re_L = 4.8 \times 10^6$, the data in Figure 6 of Reference 16 indicate that C_y was about $-.0075$. This value is about 24 times as large as the calculated value -3.11×10^{-4} . The experimental value of C_N was about .148 (Figure 11 of Reference 16); the calculated value is .1385. The calculated value is 93 percent of the experimental value. Slender-body theory, without any boundary layer, p. 67 of Reference 6, gives the value 2α or .1396. This is almost equal to the calculated value with boundary layer.

In Reference 17 the results of tests of a 4.9 caliber smooth body are presented. The nose is an almost flat surface of about .198 calibers; this is followed by a 16-degree half-angle cone for about .775 calibers; this in turn is followed by a tangent ogive for the next 2.5 calibers. The rear portion is a cylinder. The Reynolds number based on body length was about 3.4×10^6 ; at this Reynolds number the shape of the nose portion almost certainly ensures turbulent flow rearward from just behind the nose cap. For this body at a Mach number of .8, the lowest tested, the value of C_y was found to be about -.0125. The change in body shape and Mach number from the tests of Reference 16 to those of Reference 17 resulted in a change in C_y from -.0075 to -.0125. The value computed for the half-ellipsoid is -3.11×10^{-4} . Consequently it appears that the calculated value of C_y for the half-ellipsoid is probably much too small. The half-ellipsoid may have a more favorable pressure distribution than the bodies tested and thus a thinner boundary layer. If so, it would have a smaller Magnus force. It is not, however, likely that this is the explanation of the large difference between the measured and calculated values of C_y .

The calculated value of C_n , the Magnus yawing moment coefficient, Equation (154), is 1.058×10^{-3} . The value of C_n calculated by use of Figure 7 of Reference 16 is about -1.7×10^{-3} . The experimental values become zero at $\alpha = 6^\circ$ and positive for larger values of α . No values are shown for $\alpha < 4^\circ$. The value of C_n for the body of Reference 17 (Figure 42) is about 1.61×10^{-2} .

DISCUSSION OF METHOD

The present method computes a displacement surface by the use of boundary-layer theory. Consequently all the boundary layer approximations are inherent in the method. In order for boundary-layer theory to be applicable the boundary-layer thickness at a point on the surface should be much smaller than the distance from the point to the stagnation point. In the calculation for the half-ellipsoid the largest δ occurs at $x=0$, $\theta=0$. From Figure 11 it appears that δ is about 25 percent of r_0 at $x=0$, $\theta=0$. Although δ is a large fraction of r_0 , it is less than 2.5 percent of the distance to the stagnation point (see Figure 9). Therefore the boundary layer is thin as required by boundary-layer theory.

When the boundary layer approximations are made, it can be shown that

$$\frac{\partial P}{\partial y} = \frac{\rho v^2}{r_0} \cos \phi$$

When v is of the same order as u ($u = o(1)$), $\frac{\partial P}{\partial y}$ is of order unity instead of $o(\delta)$ as is usual without spin ($\delta \ll 1$). Therefore the static pressure difference across the boundary layer, which is

proportional to $\frac{\partial P}{\partial y}$ can be of $o(\delta)$ for a body with spin instead of

$o(\delta^2)$ as is usual without spin. Although the difference in static pressure across the boundary layer is of $o(\delta)$ instead of $o(\delta^2)$, it is neglected in the present analysis because terms of $o(\delta)$ have been neglected in deriving the boundary layer equations from the complete equations of motion and because its inclusion would have made the method more complicated. Kelley and Thacker in their analysis for a cylinder with a laminar boundary layer find that the effect of the pressure gradient across the boundary layer changes the Magnus force by about 30 percent. Although this is appreciable, for the half-ellipsoid an effect of this amount would not change the impression that the calculated value of the Magnus force coefficient is much too small.

In the present method, the velocity profile shape parameter n is fixed before a calculation is made and does not vary along a streamline. If n were allowed to change by introducing an additional equation for the change of n , the displacement thickness δ^* would respond more accurately to pressure gradients along the streamlines. The introduction of an equation for the change in n would, however, further complicate an already complicated method and introduce additional uncertainties associated with information needed for the n equation.

The assumption that the u and v velocity profiles have the same δ has been shown by Cooke (Reference 18) to be a source of inaccuracy. The use of two δ 's would introduce another dependent variable and a partial differential equation for it. The feeling is that instead of introducing an equation for n and an equation for another δ it might be better to investigate the use of a finite difference method instead of the integral approach. Of course this introduces other problems, one of which is the effect on the turbulence of a spinning wall. Although this seems not to appear in the present analysis, it is hidden in the friction formula and velocity profile shape, both of which involve assumptions.

The use of slender-body theory, a theory for very thin bodies at very small angle of attack, to calculate the Magnus force further decreases the accuracy of the results. At first glance, it appears that more accurate results can be obtained by using the method of Reference 7 and 8. Although slender-body theory requires no information for the wake, the method of Reference 7 and 8 cannot be used unless the distribution of the displacement thickness behind the blunt base is known. The method of Reference 7 and 8 cannot be used to calculate the flow near the base of a body with a blunt base and zero wake displacement thickness; infinite velocities are obtained at the corner. Because the behavior of the wake is unknown and because the Magnus force is small, the inaccuracy caused by an inexact representation of the wake can be important. For a body with spin and angle of attack the wake is not axisymmetric. It seems that in order to get an accurate solution it may be necessary to solve the entire flow problem, boundary layer plus wake.

Another source of error is the lack of a reliable method for calculating where on the body the initial laminar boundary layer changes to a turbulent one. Consequently unless the transition point is fixed or known in some way, the initial conditions for the turbulent boundary layer and the location of its origin are also unknown. Therefore even if the entire flow field could be solved for exactly, the results might still be unreliable.

For tests of analytical methods for computing Magnus forces it would be desirable to test a simple shape such as a half-ellipsoid at a low Mach number. The advantage of a shape like a half-ellipsoid is that the potential flow velocity distribution over at least the forward portion is given by an analytic expression. This is a help in obtaining velocity derivatives and in finding the flow near the stagnation point at angle of attack.

REFERENCES

1. Martin, J. C., "On Magnus Effects Caused by the Boundary-Layer Displacement Thickness on Bodies of Revolution at Small Angles of Attack," Journal of the Aeronautical Sciences, Vol. 24, No. 6, p. 421, Jun 1957
2. Platou, A. S., "Magnus Characteristics of Finned and Nonfinned Projectiles," AIAA Journal 3, 83-90 (1965)
3. Kelly, H. R. and Thacker, G. R., "The Effect of High Spin on the Magnus Force on a Cylinder at Small Angles of Attack," Naval Ordnance Test Station, NAVORD Report 5036 (Feb 1956)
4. Sedney, R., "Laminar Boundary Layer on a Spinning Cone at Small Angles of Attack in a Supersonic Flow," Journal of the Aeronautical Sciences, Vol. 24, No. 6, p. 430, Jun 1957
5. Ward, G. N., "Linearized Theory of Steady High-Speed Flow," Cambridge Univ. Press (1955)
6. Nielsen, J. N., "Missile Aerodynamics," McGraw-Hill (1960)
7. Dawson, C. W., and Dean, J. S., "The XYZ Potential Flow Program," Tech. Note CMD-22-71, Jun 1971, NSRDC Bethesda, MD.
8. Hess, J. I. and Smith A. M. O., "Calculation of Potential Flow About Arbitrary Bodies," Progress in Aeronautical Sciences Vol. 8, Pergamon Press (1967)
9. Lighthill, M. J., "On Displacement Thickness," Journal of Fluid Mechanics, Vol. 14, Part 4, p. 383, Aug 1958
10. Rosenhead, L. (ed.), "Laminar Boundary Layers," Oxford Univ. Press (1963)
11. Theodorsen, T., "Theory of Wing Sections of Arbitrary Shape," NACA TR 411 (1931).
12. Theodorsen, T., and Garrick, I. E., "General Potential Theory of Arbitrary Wing Sections," NACA TR 452, (1933)
13. Millikan, C. B., "An Extended Theory of Thin Airfoils and its Application to the Biplane Problem," NACA TR 362, (1930)
14. Milne-Thomson, L. M., "Theoretical Hydrodynamics 5th ed. Macmillan, New York, (1968)
15. Lamb, H., "Hydrodynamics," 6th ed. Dover, New York (1945)

16. Carman, J. B., Uselton, J. C., and Summers, W. E., "Experimental Magnus Characteristics of Basic and Boattail Configurations of 3- and 5-Cal. Army-Navy Spinner Projectiles at Subsonic and Transonic Mach Numbers," AEDC-TR-70-36, Apr 1970
17. Regan F. J. Jr., and Schermerhorn, V. L., "Aeroballistic Evaluation and Computer Stability Analysis of the U. S. Navy 20-Millimeter General Purpose Projectile," NOLTR 71-95, Jun 1971
18. Cooke, J. C., "Pohlhausen's Method for Three-Dimensional Laminar Boundary Layers," Aeronaut. Quart. 3, 51-60
19. Shapiro, A. H., "The Dynamics and Thermodynamics of Compressible Fluid Flow," Vol. I Ronald Press, New York, (1953)
20. Pennington, R. H., "Introductory Computer Methods and Numerical Analysis," Macmillan Co., New York, (1965)

----- BOUNDARY LAYER DISPLACEMENT SURFACE

\bar{F}_Y MAGNUS FORCE

\bar{M}_Y MAGNUS MOMENT ABOUT NOSE

$\bar{\omega}$ ANGULAR SPIN VELOCITY

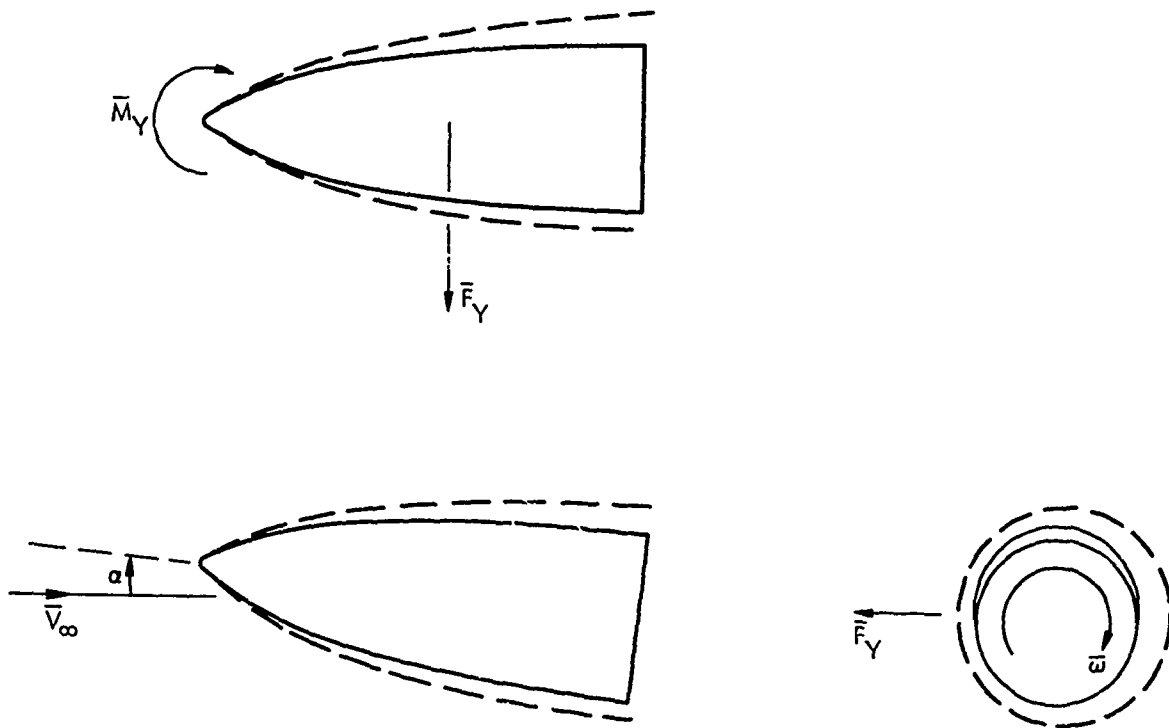


FIG. 1 FLOW ABOUT SPINNING AXISYMMETRIC BODY AT SMALL ANGLE OF ATTACK

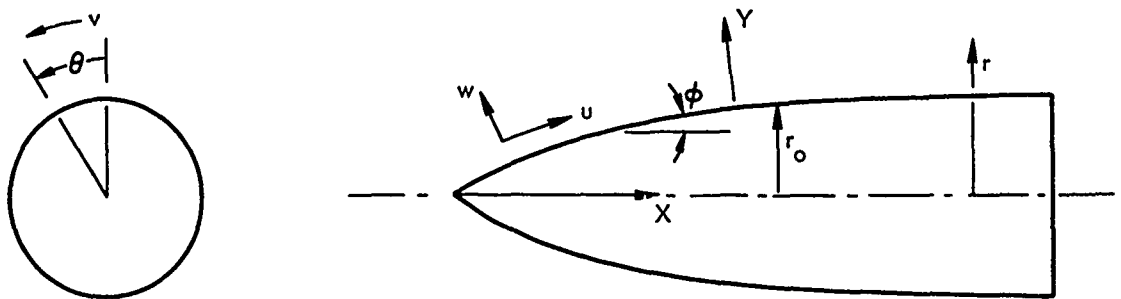


FIG. 2 COORDINATE SYSTEM

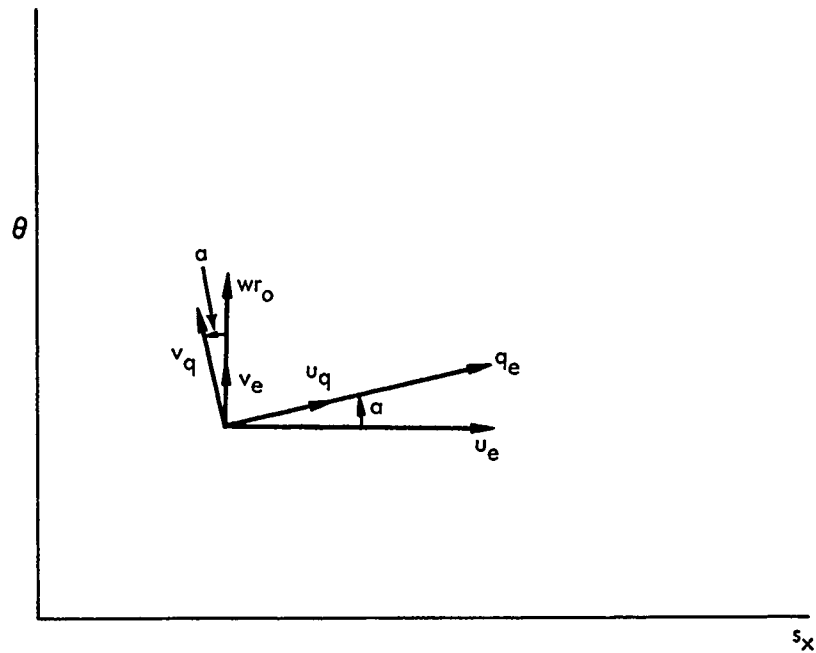


FIG. 3 VELOCITIES

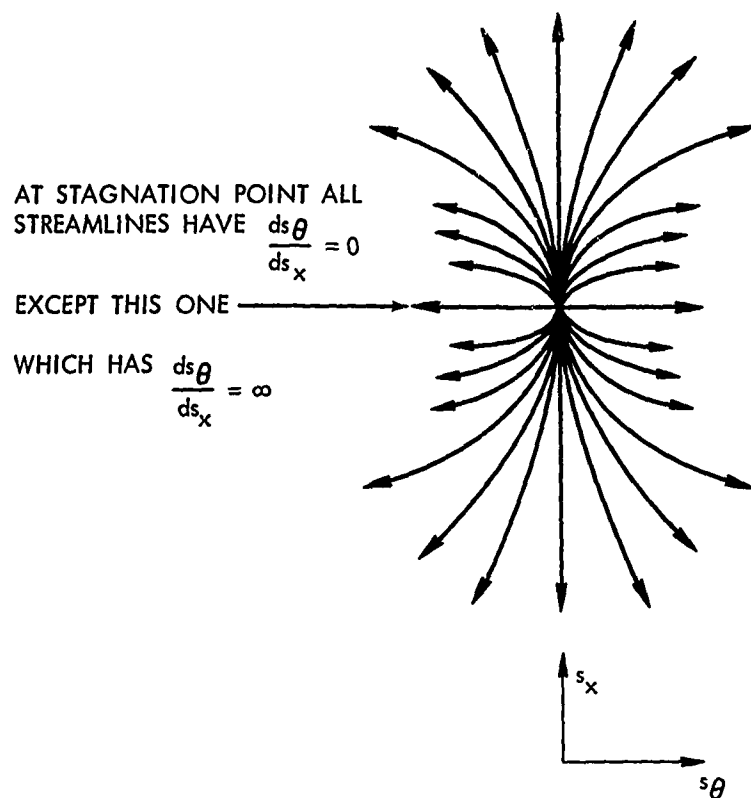


FIG. 4 STREAMLINES AT STAGNATION POINT

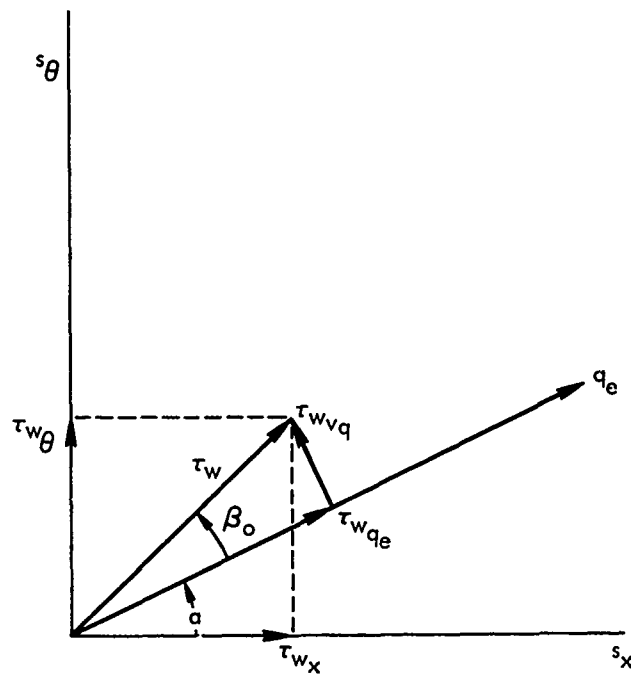


FIG. 5 SHEAR STRESS COMPONENTS

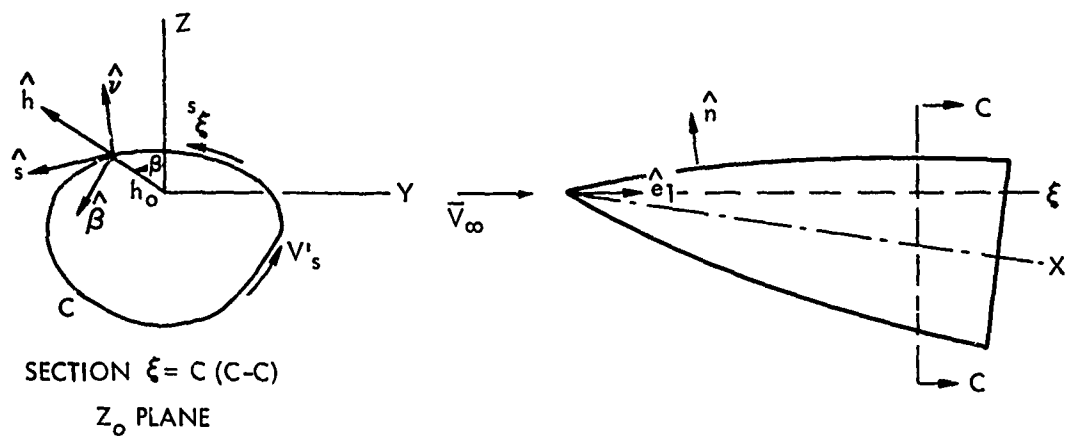


FIG. 6 COORDINATES FOR APPLICATION OF SLENDER-BODY THEORY

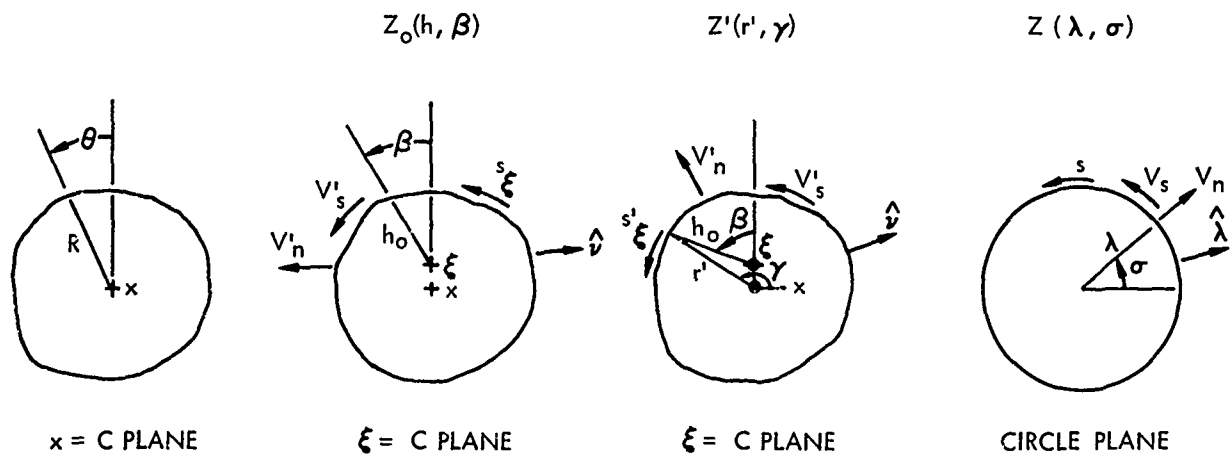


FIG. 7 CROSS SECTIONS USED TO FIND V'_s FOR USE IN SLENDER-BODY THEORY.

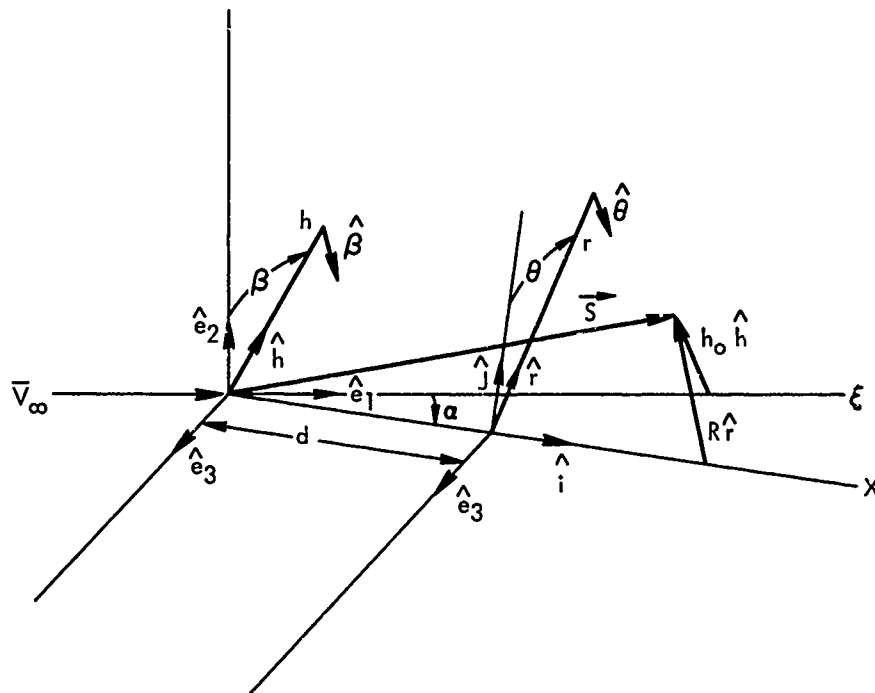


FIG. 8 COORDINATE SYSTEMS

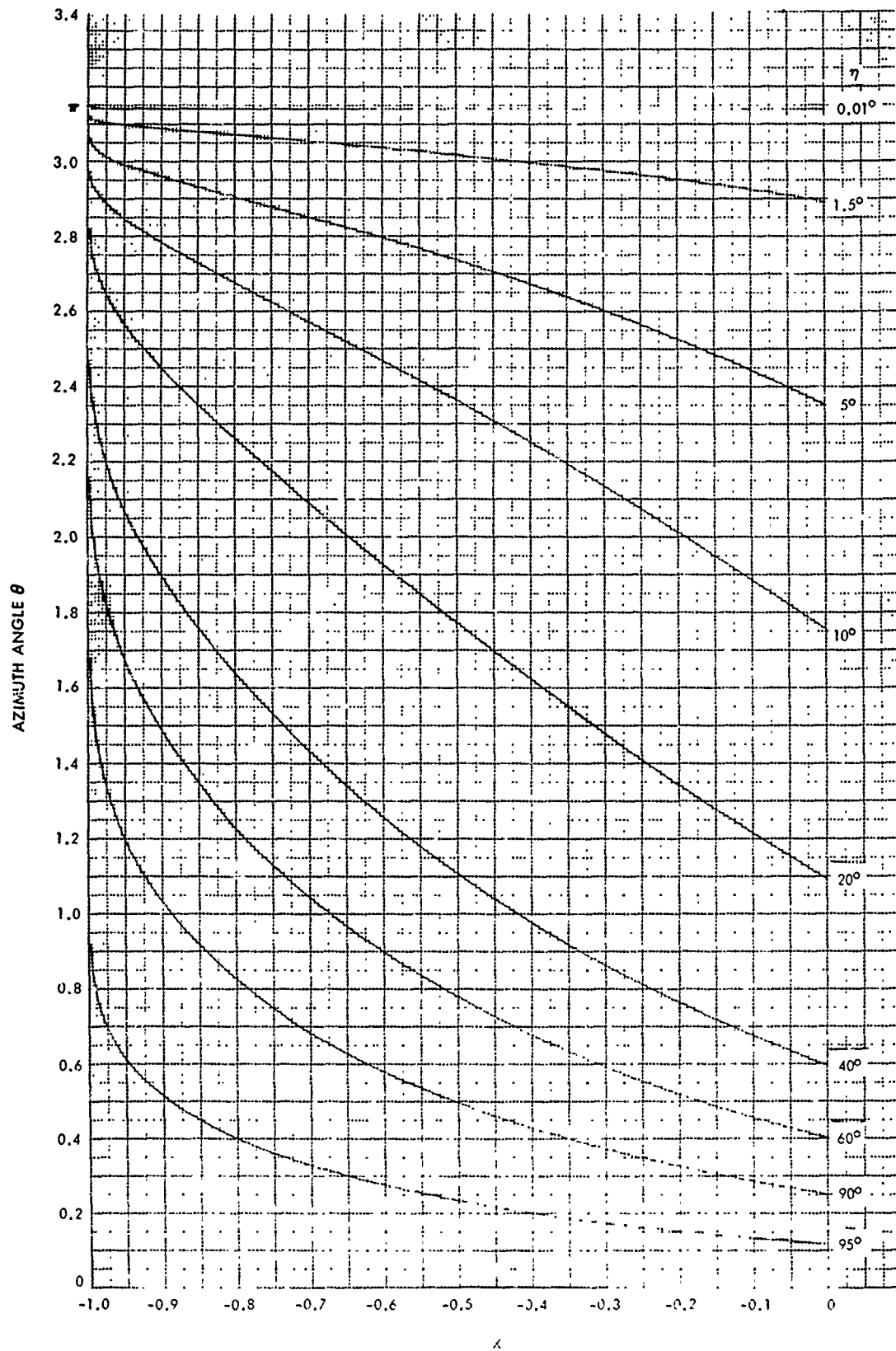


FIG. 9 STREAMLINES ON FINENESS RATIO 5 HALF-ELLIPSOID AT 4 DEGREES ANGLE OF ATTACK FOR VARIOUS STREAMLINE STARTING ANGLES η

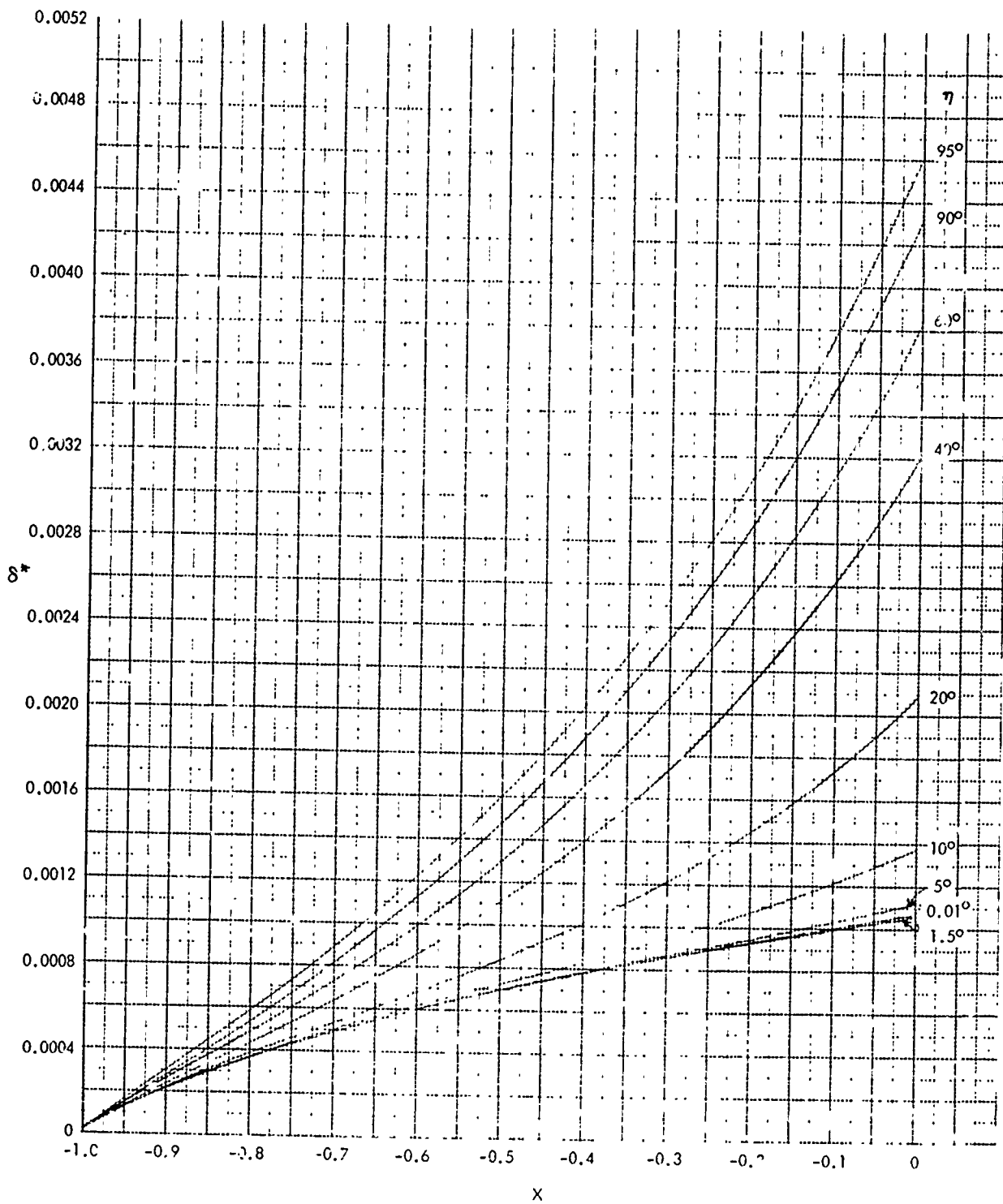


FIG. 10a BOUNDARY LAYER DISPLACEMENT THICKNESS δ^* ON HALF-ELLIPSOID ON VARIOUS STREAMLINES SPECIFIED BY STARTING ANGLE ($\eta > 0$).

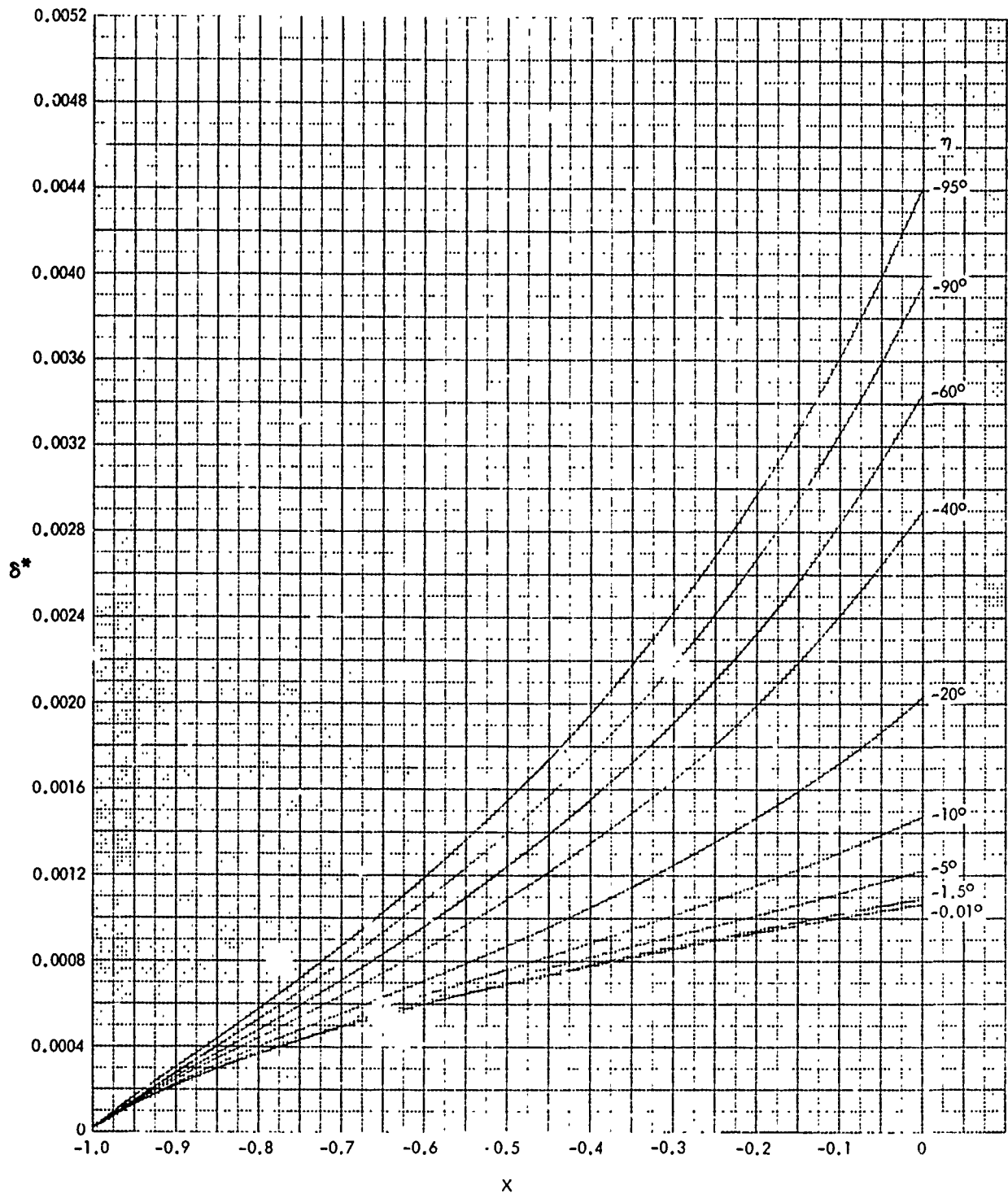


FIG. 10b BOUNDARY LAYER DISPLACEMENT THICKNESS δ^* ON HALF-ELLIPSOID ON VARIOUS STREAMLINES SPECIFIED BY STARTING ANGLE η ($\eta < 0$).

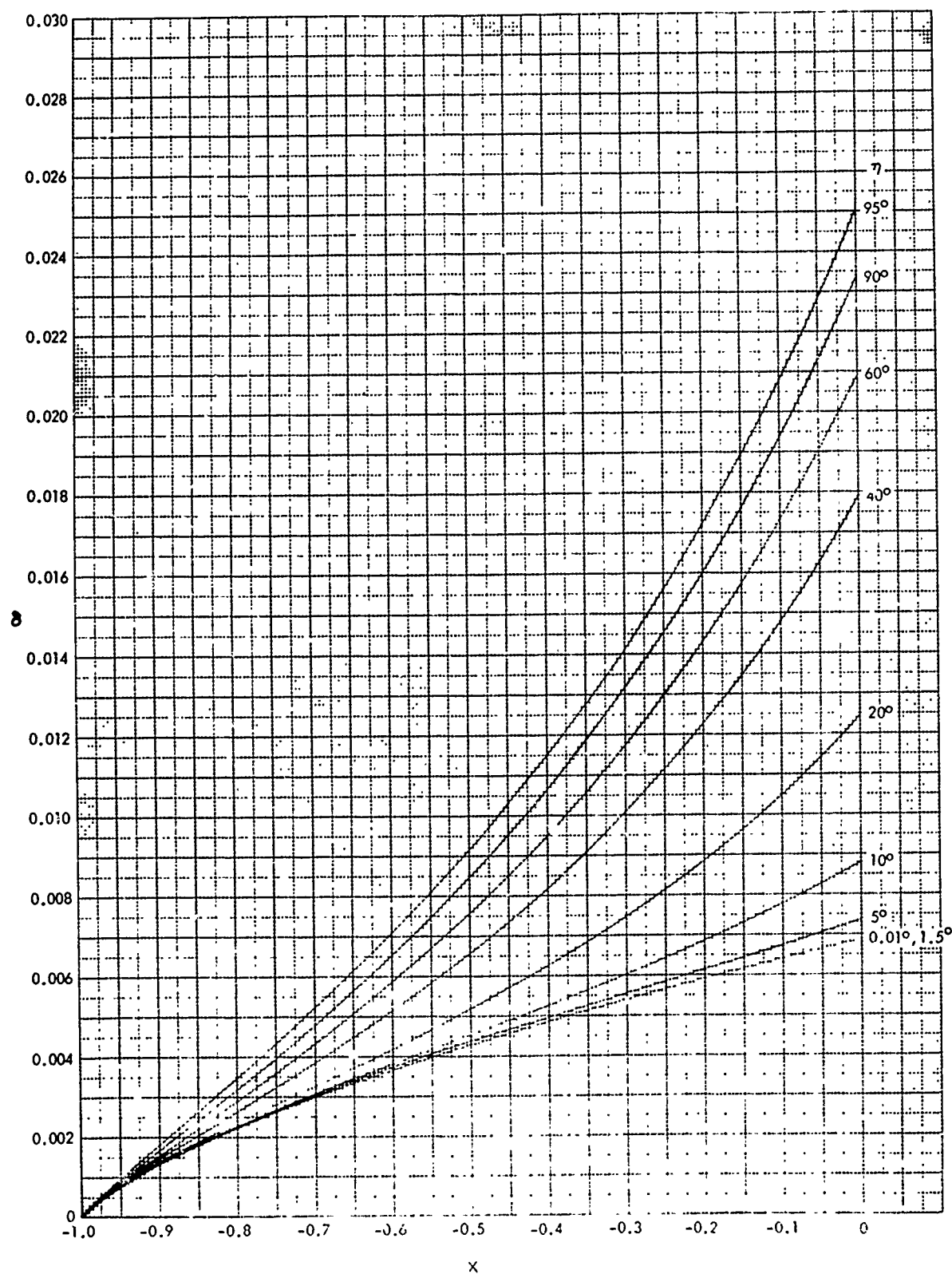


FIG. 11a BOUNDARY LAYER THICKNESS δ ON HALF-ELLIPSOID ON VARIOUS STREAMLINES SPECIFIED BY STARTING ANGLE η ($\eta > 0$).

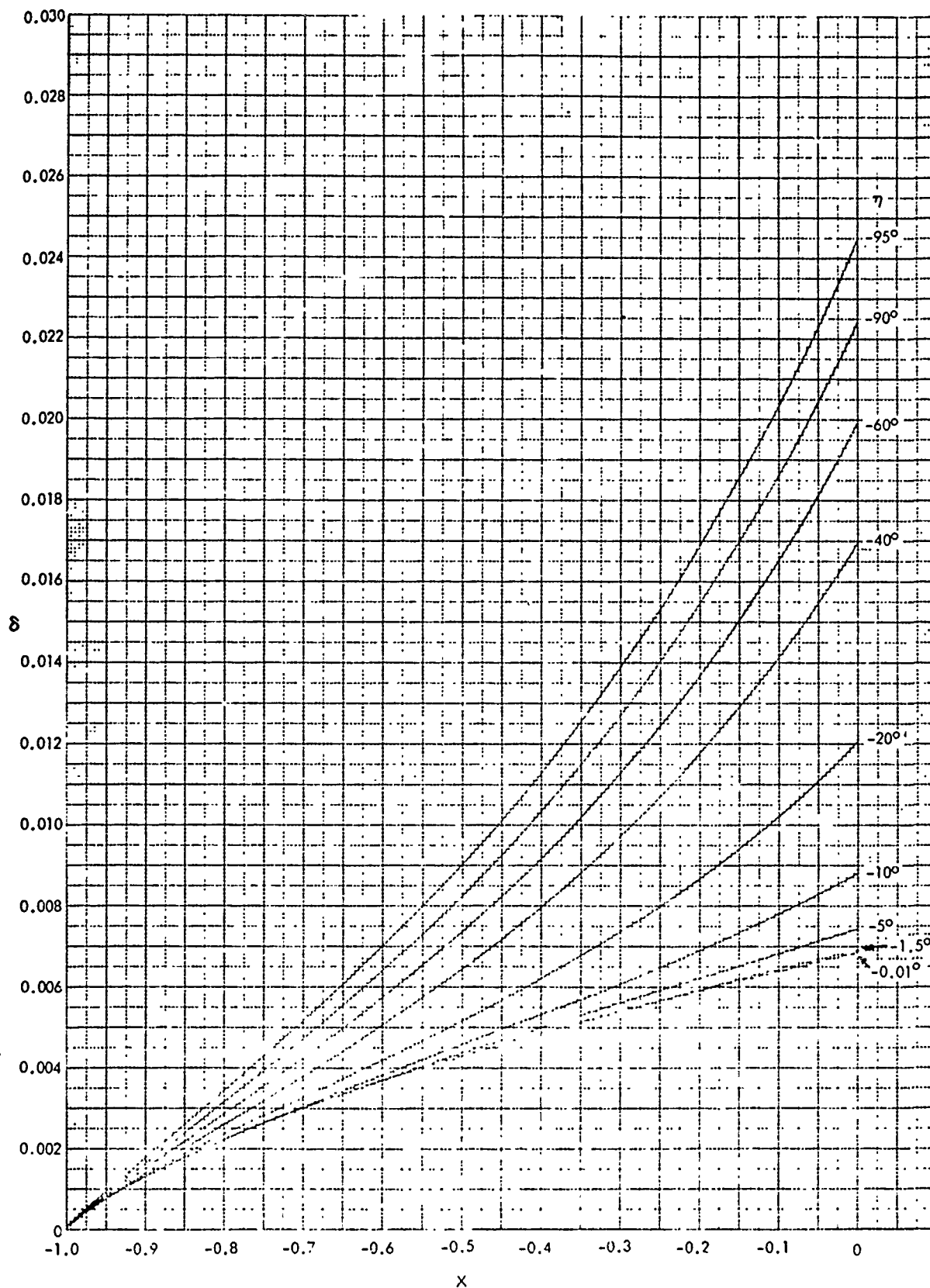


FIG. 11b BOUNDARY LAYER THICKNESS δ ON HALF-ELLIPSOID ON VARIOUS STREAMLINES SPECIFIED BY STARTING ANGLE η ($\eta < 0$).

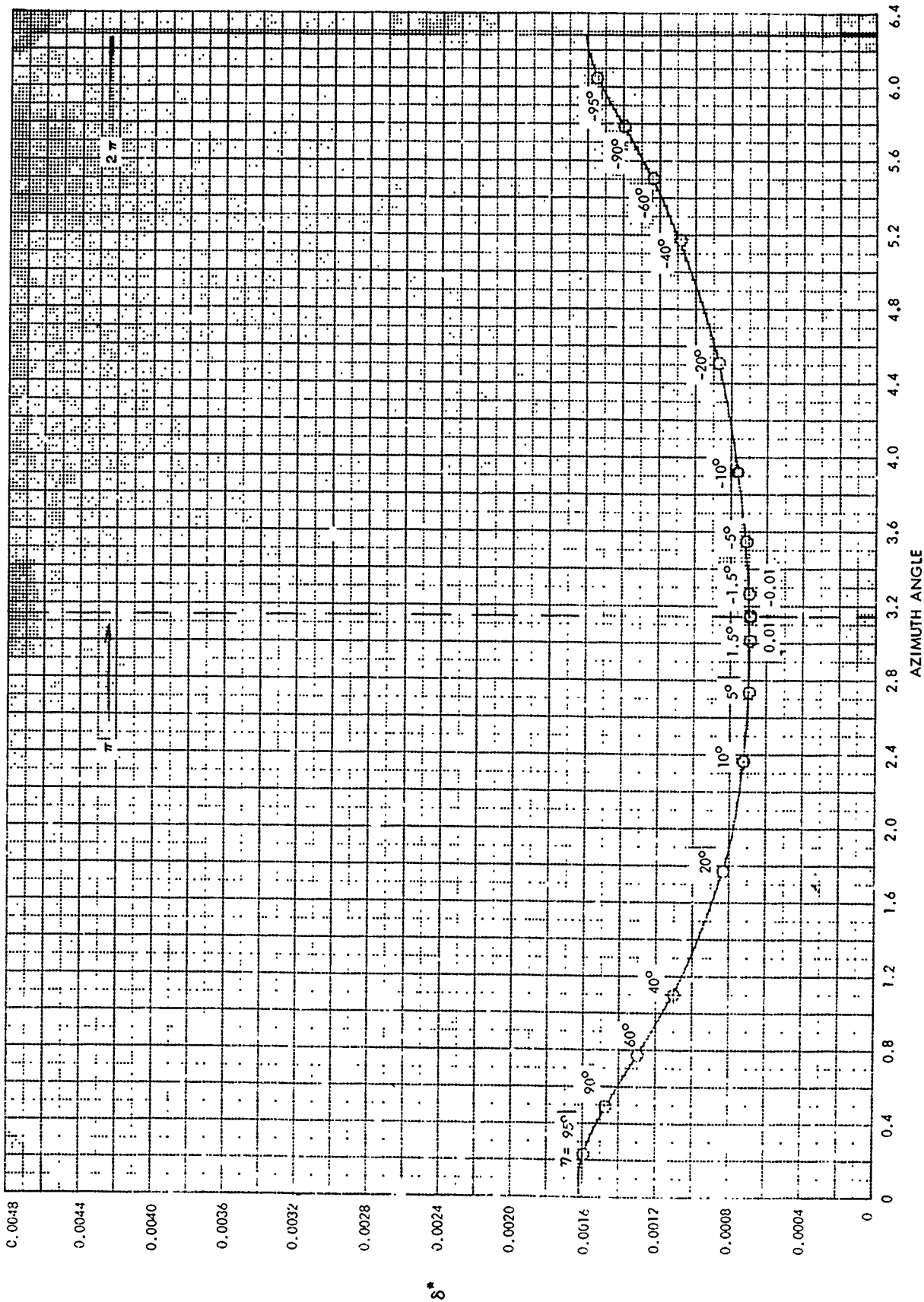


FIG. 12a BOUNDARY LAYER DISPLACEMENT THICKNESS δ^* VERSUS AZIMUTHAL ANGLE θ AT $X = -0.5$.

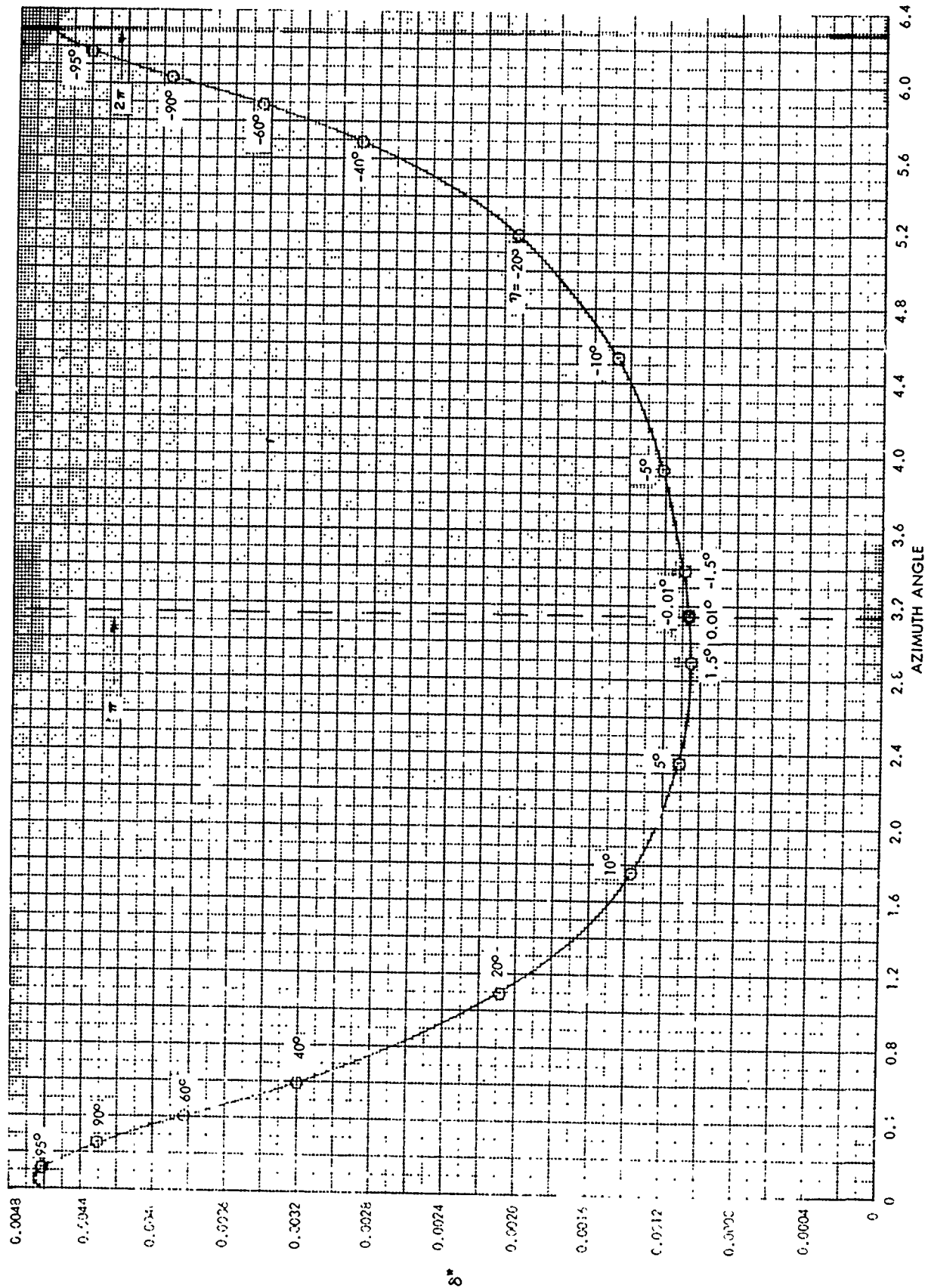


FIG. 12b BOUNDARY LAYER DISPLACEMENT THICKNESS δ^* VERSUS AZIMUTHAL ANGLE θ AT $x = 0$.

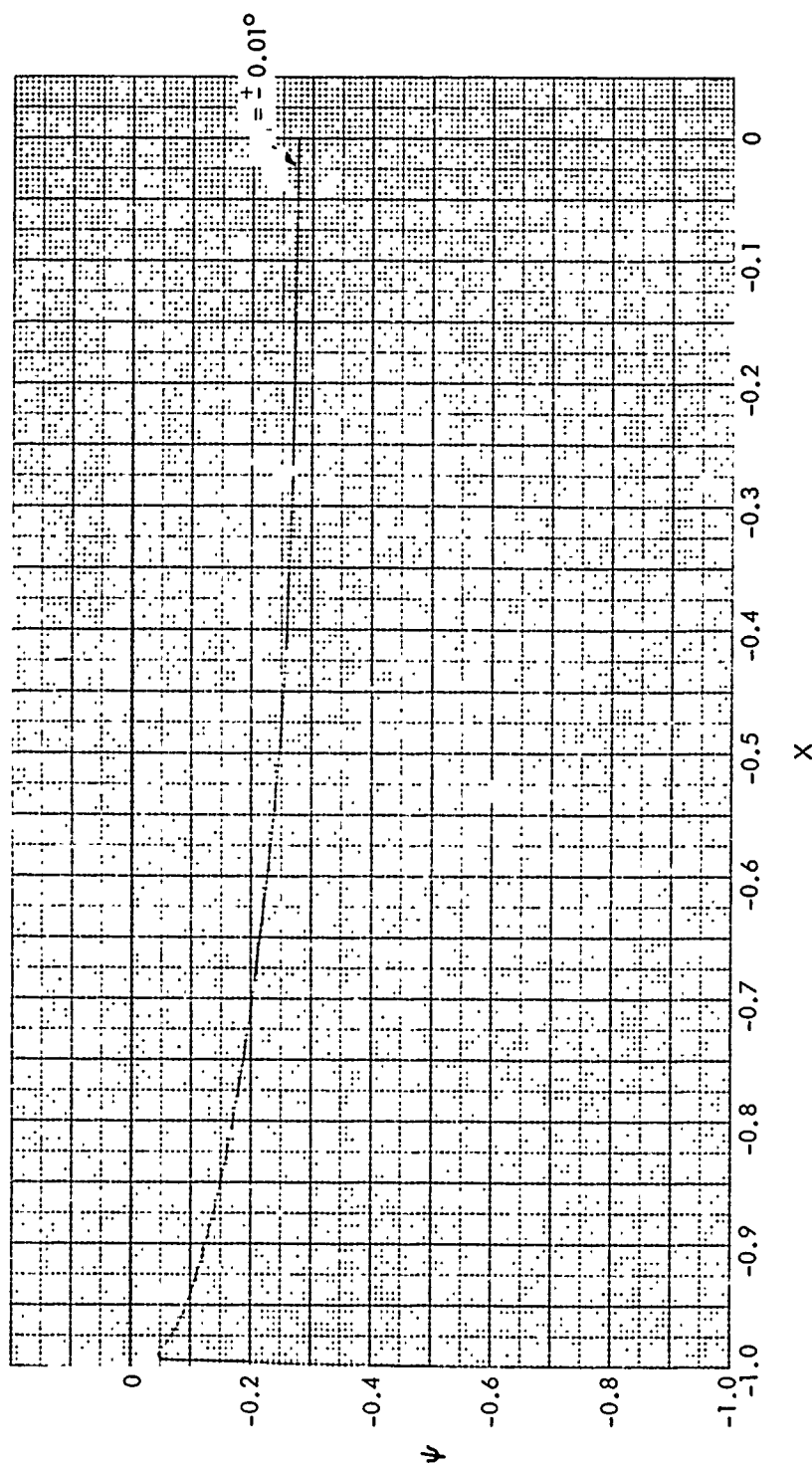


FIG. 13a ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS
FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 0.01^\circ$.

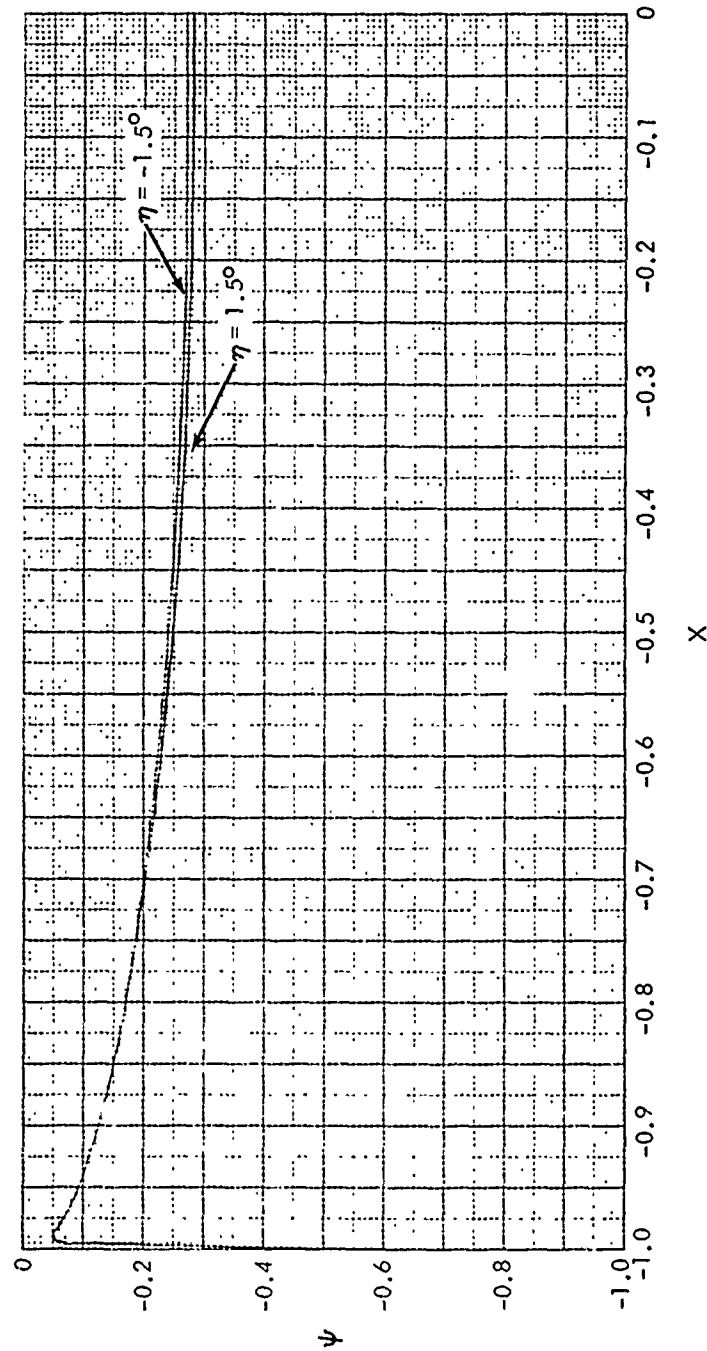


FIG. 13b ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS
FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 1.5^\circ$.

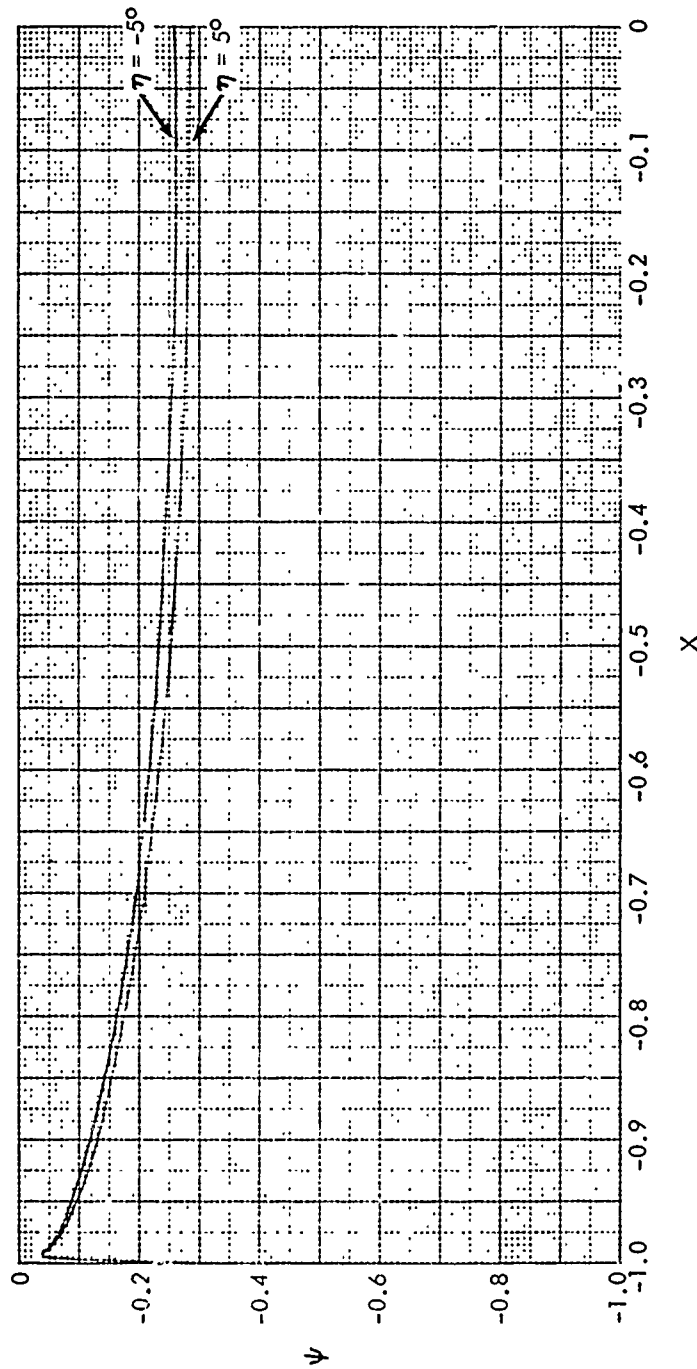


FIG. 13c ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS
FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 50^\circ$.

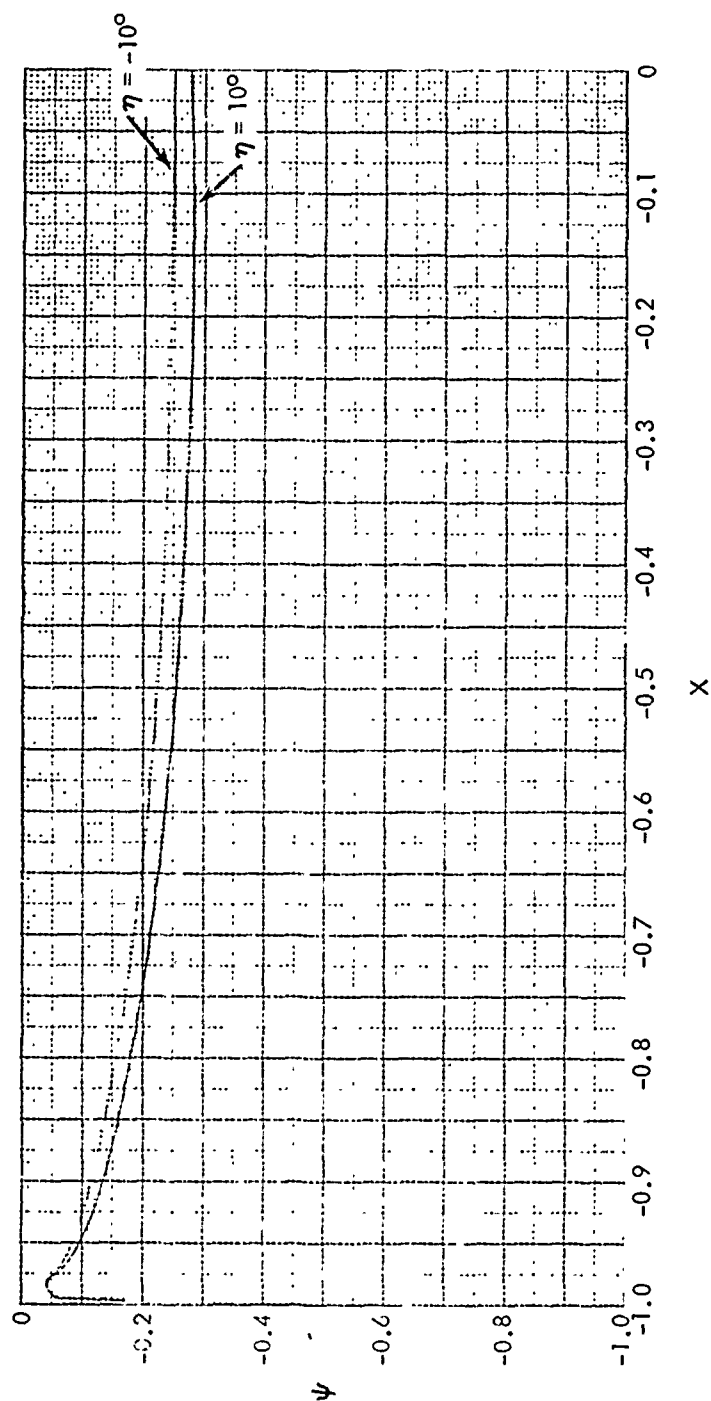


FIG. 13d ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 10^\circ$.

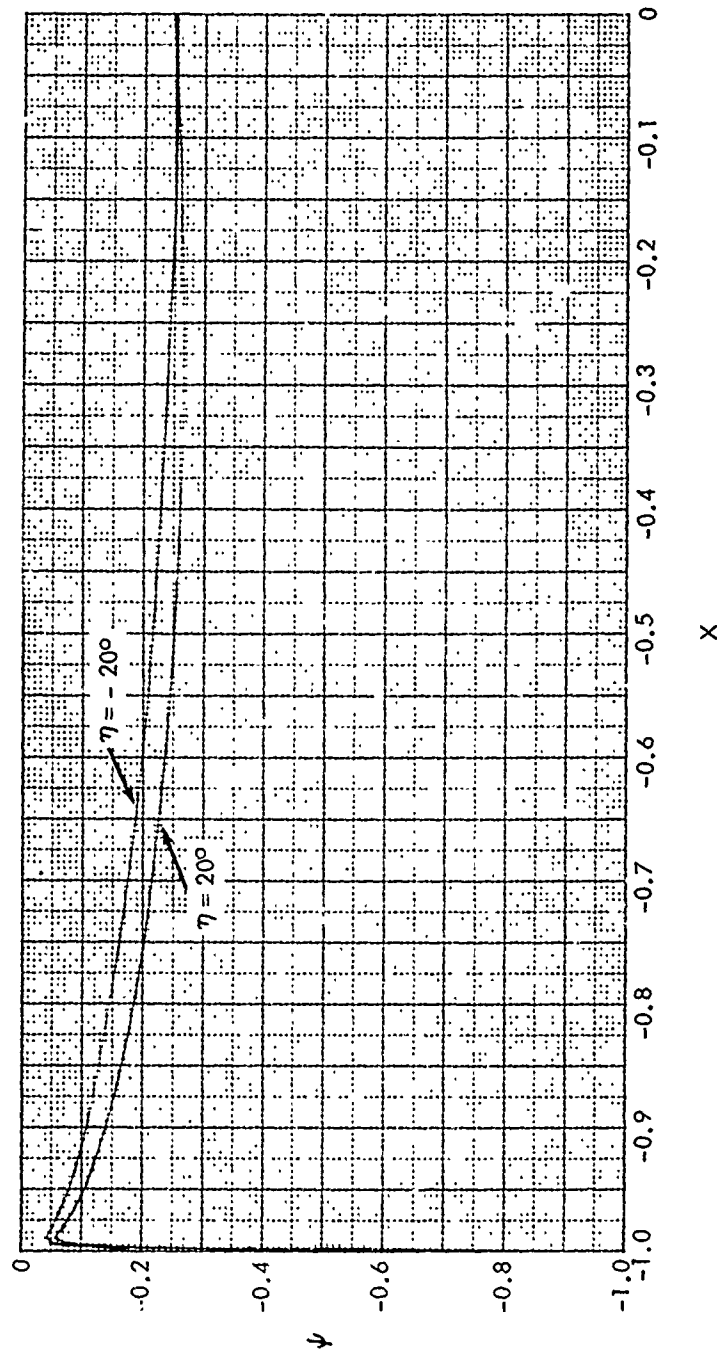


FIG. 13e ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 20^\circ$.

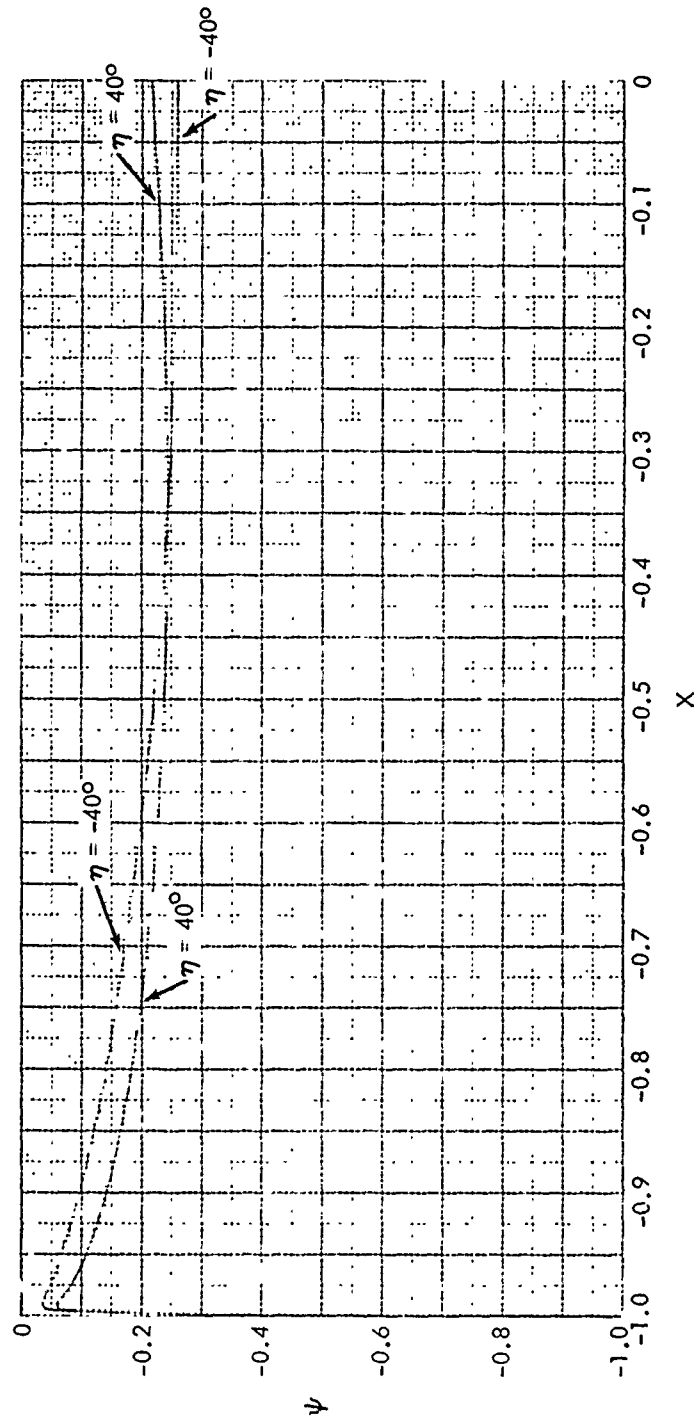


FIG. 13f ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS
FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 40^\circ$.

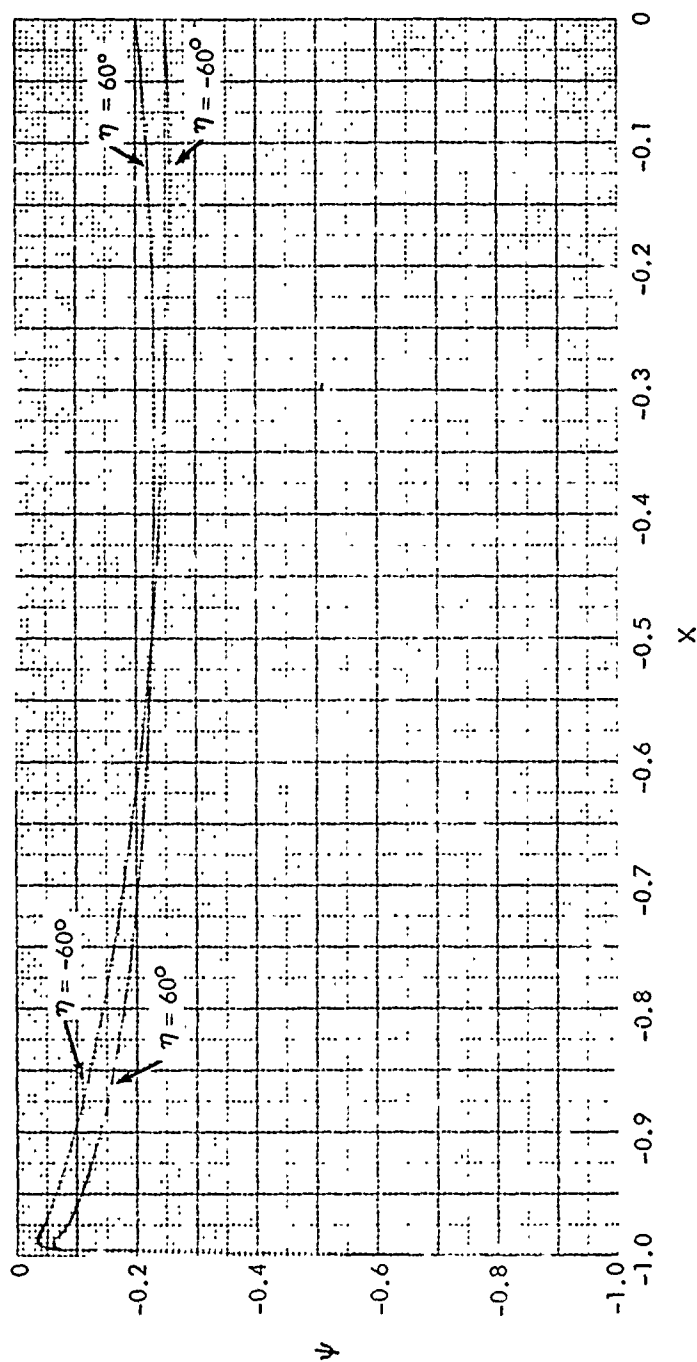


FIG. 13g ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 60^\circ$.

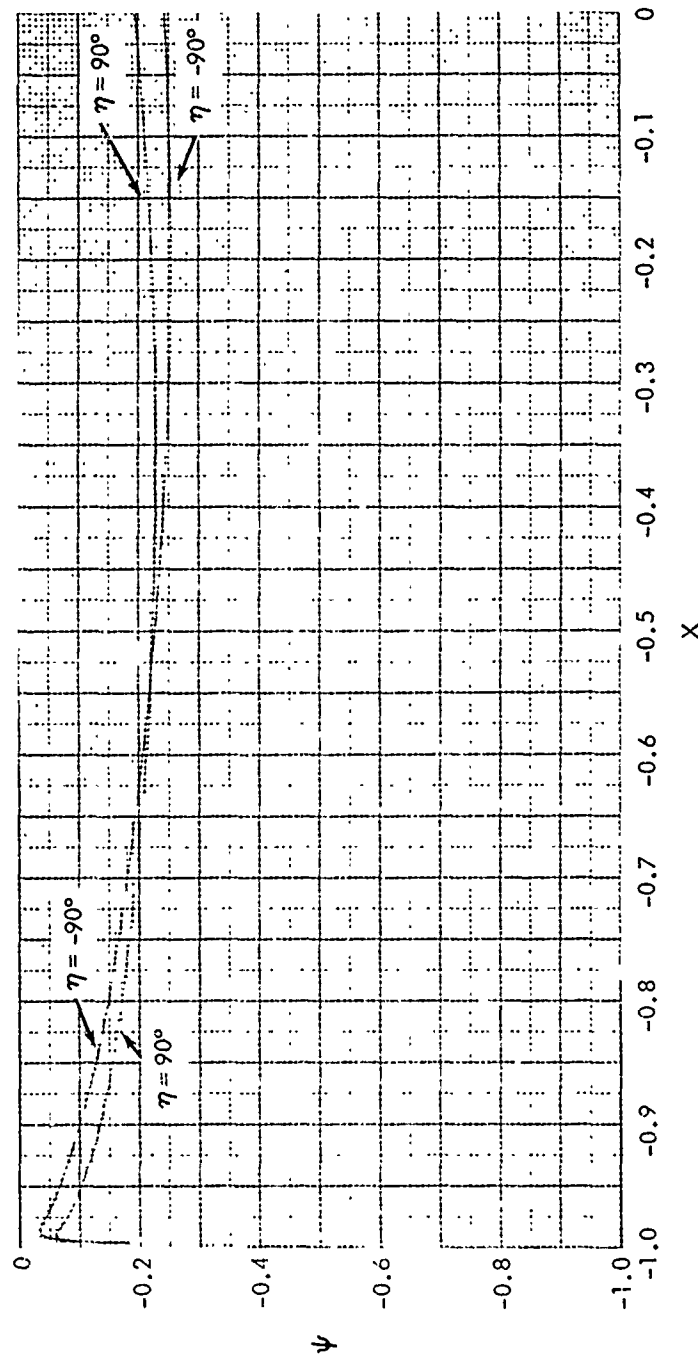


FIG. 13h ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 90^\circ$.

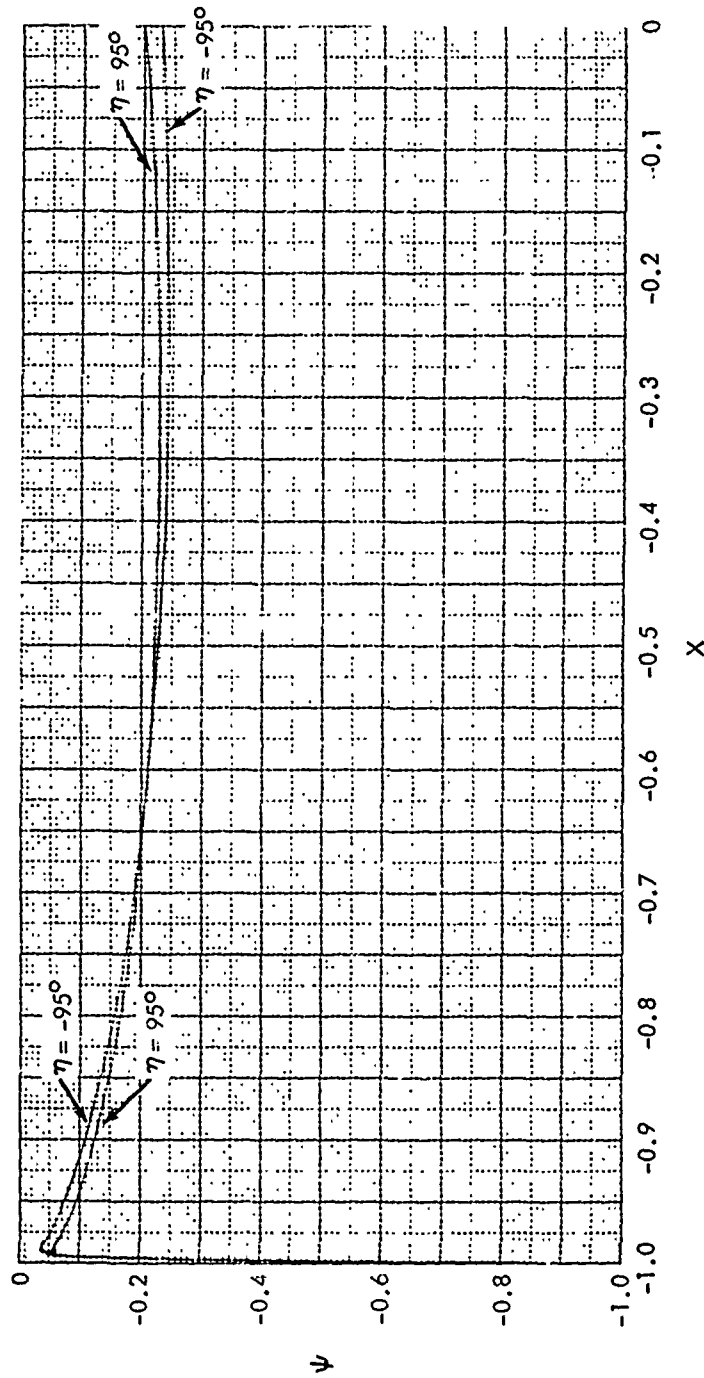


FIG. 13i ψ , TANGENT OF ANGLE BETWEEN POTENTIAL FLOW STREAMLINE AND SURFACE SHEAR STRESS FOR STREAMLINES WITH STARTING ANGLES $\eta = \pm 95^\circ$.

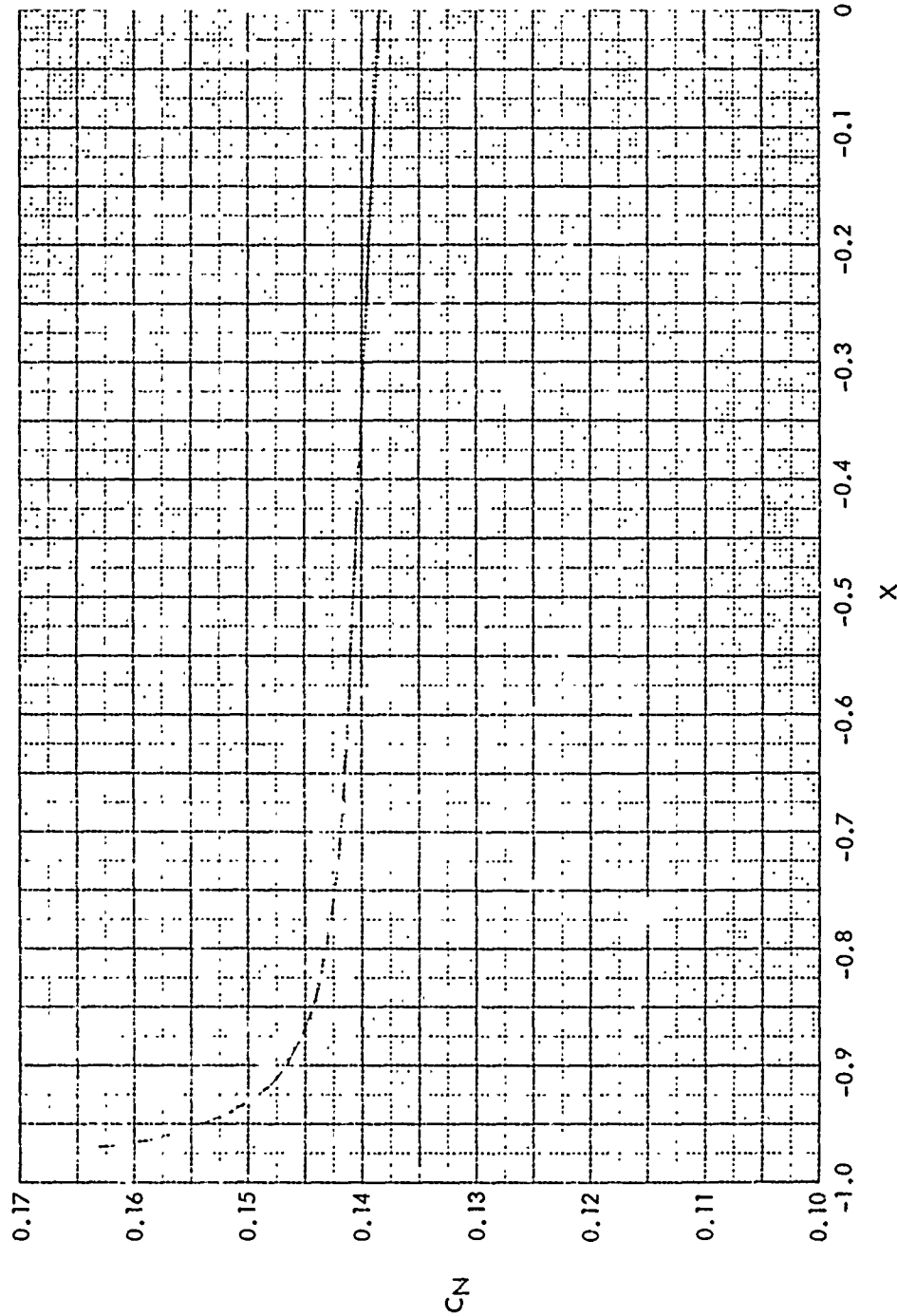


FIG. 14 NORMAL FORCE COEFFICIENT C_N BASED ON LOCAL CROSS SECTION AREA FOR PORTION OF BODY BETWEEN STATION X AND THE BODY NOSE.

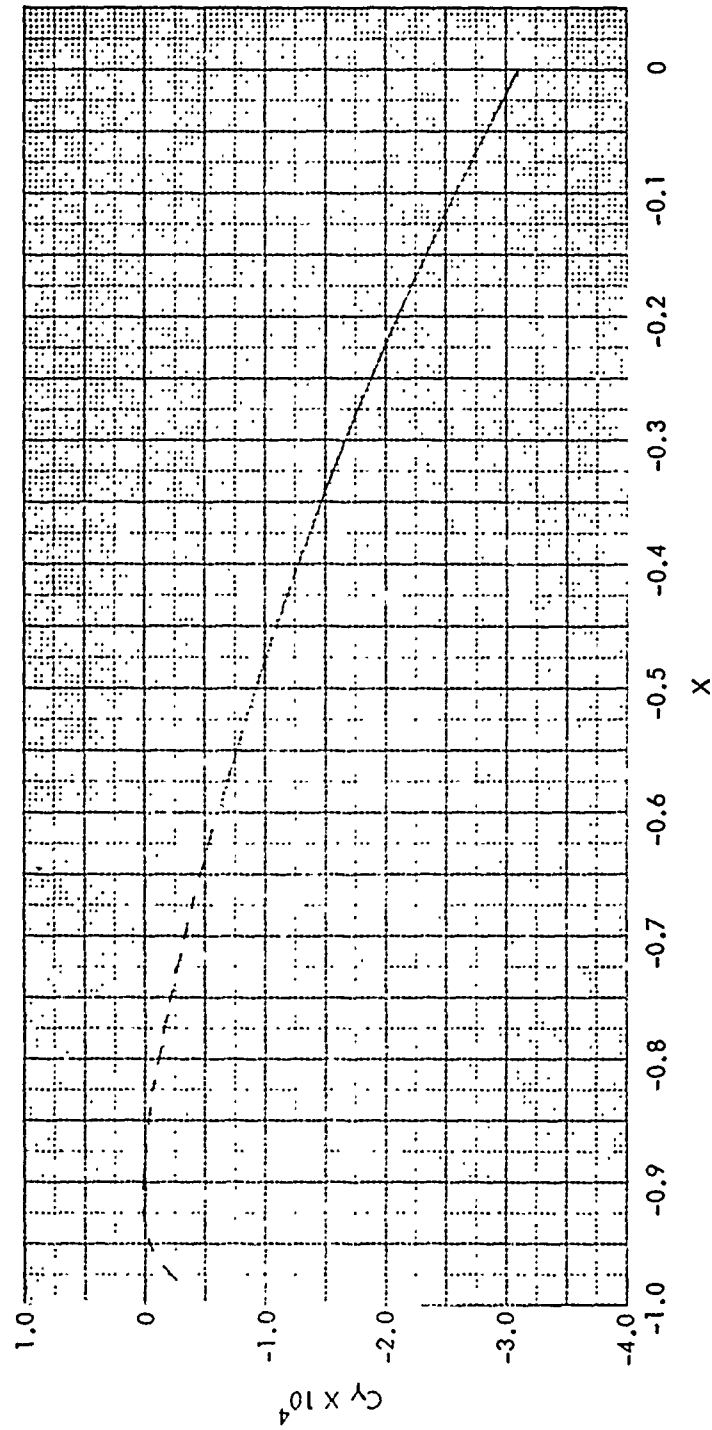


FIG. 15 MAGNUS FORCE COEFFICIENT C_y BASED ON LOCAL CROSS SECTION AREA FOR PORTION OF BODY BETWEEN STATION X AND BODY NOSE.

APPENDIX A

To obtain an expression for $\frac{\partial a}{\partial x}$ at the stagnation point for $\alpha \neq 0$, the expression

$$\tan a = \frac{v_e}{u_e} \quad (A-1)$$

is used. By symmetry, the stagnation point lies on the curve $\theta = \pi$ and $v_e = 0$ on $\theta = \pi$ for all x . Therefore, $a = 0$ for $\theta = \pi$ for all x ; therefore, $\frac{\partial a}{\partial x} = 0$ on $\theta = \pi$ and therefore at the stagnation point.

Also $\frac{\partial v}{\partial x} = 0$ for all x so that the term

$$\frac{1}{u_e} \frac{\partial v_e}{\partial x} G_7 r_o \cos \phi \quad (A-2)$$

in (86) is zero for $\theta = \pi$.

To obtain an expression for $\frac{\partial a}{\partial \theta}$ at the stagnation point, use (A-1) to get

$$\frac{\partial a}{\partial \theta} = \frac{u_e \frac{\partial v_e}{\partial \theta} - v_e \frac{\partial u_e}{\partial \theta}}{u_e^2 + v_e^2} \quad (A-3)$$

For $x = x_s$ and θ near π ,

$$u_e = \left(\frac{\partial u_e}{\partial \theta} \right)_s \Delta \theta + \left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s \frac{\Delta \theta^2}{2} + \dots$$

$$v_e = \left(\frac{\partial v_e}{\partial \theta} \right)_s \Delta \theta + \left(\frac{\partial^2 v_e}{\partial \theta^2} \right)_s \frac{\Delta \theta^2}{2} + \dots$$

$$\frac{\partial u_e}{\partial \theta} = \left(\frac{\partial u_e}{\partial \theta} \right)_s + \left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s \Delta \theta + \dots$$

$$\frac{\partial v_e}{\partial \theta} = \left(\frac{\partial v_e}{\partial \theta} \right)_s + \left(\frac{\partial^2 v_e}{\partial \theta^2} \right)_s \Delta \theta + \dots$$

Because u_e is symmetric with respect to $\theta = \pi$ it follows that $\left(\frac{\partial u_e}{\partial \theta} \right)_s = 0$ for $\theta = \pi$; therefore, $\left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s \neq 0$ for $\alpha \neq 0$. Moreover, v_e is antisymmetric with respect to $\theta = \pi$. Therefore, $\left(\frac{\partial^2 v_e}{\partial \theta^2} \right)_s = 0$ for $\theta = \pi$; thus $\left(\frac{\partial v_e}{\partial \theta} \right)_s \neq 0$ for $\alpha \neq 0$. Then (A-3) becomes

$$\frac{\partial a}{\partial \theta} = \frac{\left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s \frac{\Delta \theta^2}{2} \left(\frac{\partial v_e}{\partial \theta} \right)_s - \left(\frac{\partial v_e}{\partial \theta} \right)_s \Delta \theta \left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s \Delta \theta}{\left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s^2 \frac{\Delta \theta^4}{4} + \left(\frac{\partial v_e}{\partial \theta} \right)_s^2 \Delta \theta^2}$$

or

$$\frac{\partial a}{\partial \theta} = \frac{-\frac{1}{2} \left(\frac{\partial^2 u_e}{\partial \theta^2} \right) \left(\frac{\partial v_e}{\partial \theta} \right)_s}{\left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s^2 \frac{\Delta \theta^2}{4} + \left(\frac{\partial v_e}{\partial \theta} \right)_s^2} \quad (A-4)$$

As the stagnation point is approached $\Delta \theta \rightarrow 0$ and (A-4) becomes

$$\left(\frac{\partial a}{\partial \theta} \right)_s = -\frac{1}{2} \frac{\left(\frac{\partial^2 u_e}{\partial \theta^2} \right)_s}{\left(\frac{\partial v_e}{\partial \theta} \right)_s} \quad (A-5)$$

APPENDIX B

From Figure 8 the vector \vec{S} to a point on the body is written, with all lengths non-dimensionalized by the body length \bar{L} , as

$$\vec{S} = \hat{i}(d+x) + R\hat{r} \quad (\text{B-1})$$

also

$$\vec{S} = \hat{e}_1 \xi + h_o \hat{h} \quad (\text{B-2})$$

or, with (B-1) and (B-2)

$$\hat{i}(d+x) + R\hat{r} = \hat{e}_1 \xi + h_o \hat{h} \quad (\text{B-3})$$

Then

$$h_o = \hat{i} \cdot \hat{h}(d+x) + R\hat{h} \cdot \hat{r} \quad (\text{B-4})$$

The unit vector \hat{h} is given by

$$\hat{h} = \hat{e}_2 \cos \beta - \hat{e}_3 \sin \beta \quad (\text{Figure 8}) \quad (\text{B-5})$$

and the unit vector \hat{r} is given by

$$\hat{r} = \hat{j} \cos \theta - \hat{e}_3 \sin \theta \quad (\text{Figure 8}) \quad (\text{B-6})$$

Then from (B-4), (B-5), and (B-6)

$$h_o = - (d+x) \sin \alpha \cos \beta + R(\cos \alpha \cos \beta \cos \theta + \sin \beta \sin \theta) \quad (B-7)$$

The expression for β is given later on.

To find $\left(\frac{\partial h_o}{\partial \xi}\right)_\beta$ let the equation of the body plus the displacement surface be

$$H(x, R, \theta) = c \quad (B-8)$$

Also

$$x = x(\xi, h, \beta) \quad (B-9)$$

$$R = R(\xi, h, \beta) \quad (B-10)$$

$$\theta = \theta(\xi, h, \beta) \quad (B-11)$$

Then

$$dx = x_\xi d\xi + x_h dh + x_\beta d\beta \quad (B-12)$$

$$dR = R_\xi d\xi + R_h dh + R_\beta d\beta \quad (B-13)$$

$$d\theta = \theta_\xi d\xi + \theta_h dh + \theta_\beta d\beta \quad (B-14)$$

Also

$$dH = H_R dR + H_x dx + H_\theta d\theta = H_\xi d\xi + H_h dh + H_\beta d\beta \quad (B-15)$$

Then by using (B-12), (B-13), and (B-14) in the L.H.S. of (B-15) and equating the coefficients of $d\xi$, dh , and $d\beta$ there is obtained

$$H_{\xi} = H_R R_{\xi} + H_x x_{\xi} + H_{\theta} \theta_{\xi} \quad (\text{B-16})$$

$$H_h = H_R R_h + H_x x_h + H_{\theta} \theta_h \quad (\text{B-17})$$

$$H_{\beta} = H_R R_{\beta} + H_x x_{\beta} + H_{\theta} \theta_{\beta} \quad (\text{B-18})$$

From $u(\xi, h_0, \beta) = c$ it follows that

$$\left(\frac{\partial h_0}{\partial \xi}\right)_{\beta} = - \frac{H_{\xi}}{H_h} \quad (\text{B-19})$$

and

$$\left(\frac{\partial h_0}{\partial \beta}\right)_{\xi} = - \frac{H_{\beta}}{H_h} \quad (\text{B-20})$$

When (B-16), (B-17), (B-18) are used in (B-19) and (B-20) the result is

$$\left(\frac{\partial h_0}{\partial \xi}\right)_{\beta} = - \frac{H_R R_{\xi} + H_x x_{\xi} + H_{\theta} \theta_{\xi}}{H_R R_h + H_x x_h + H_{\theta} \theta_h} \quad (\text{B-21})$$

and

$$\left(\frac{\partial h_0}{\partial \beta}\right)_{\xi} = - \frac{H_R R_{\beta} + H_x x_{\beta} + H_{\theta} \theta_{\beta}}{H_R R_h + H_x x_h + H_{\theta} \theta_h} \quad (\text{B-22})$$

The quantities on the R.H.S of (B-21) and (B-22) are obtained as follows. Equation (B-8) is written as

$$H(x, R, \theta) = R - R(x, \theta) = c \quad (B-23)$$

Then

$$H_R = 1 \quad (B-24)$$

$$H_x = \left(\frac{\partial R}{\partial x} \right)_\theta \quad (B-25)$$

$$H_\theta = \left(\frac{\partial R}{\partial \theta} \right)_x \quad (B-26)$$

From $R = r_0 + \delta^*$ it follows that

$$\left(\frac{\partial R}{\partial x} \right)_\theta = \left(\frac{\partial r_0}{\partial x} \right)_\theta + \left(\frac{\partial \delta^*}{\partial x} \right)_\theta \quad (B-27)$$

Also

$$\frac{d\delta^*}{dx} = \left(\frac{\partial \delta^*}{\partial x} \right)_\theta + \left(\frac{\partial \delta^*}{\partial \theta} \right)_x \frac{d\theta}{dx} \quad (B-28)$$

Therefore

$$\left(\frac{\partial \delta^*}{\partial x} \right)_\theta = \frac{d\delta^*}{dx} - \left(\frac{\partial \delta^*}{\partial \theta} \right)_x \frac{d\theta}{dx} \quad (B-29)$$

The quantity $\frac{d\delta^*}{dx}$ is computed during the integration of the equation for the displacement surface. The quantity $\left(\frac{\partial \delta^*}{\partial \theta} \right)_x$ is found from the

computed values of δ^* at various θ for a fixed x by use of the spline-fit method of Reference 20. The quantity $\frac{d\theta}{dx}$ is the stream-line direction. Equation (B-29) is used with (B-27) to get $(\frac{\partial R}{\partial x})_{\theta}$.

For $H_{\theta} = (\frac{\partial R}{\partial \theta})_x$ there is used

$$(\frac{\partial R}{\partial \theta})_x = (\frac{\partial r_o}{\partial \theta})_x + (\frac{\partial \delta^*}{\partial \theta})_x \quad (B-30)$$

where $(\frac{\partial r_o}{\partial \theta})_x = 0$ because r_o is independent of θ .

From (B-3), (B-5), and (B-6) it follows that

$$R = \xi \sin \alpha \cos \theta + h_o (\cos \alpha \cos \beta \cos \theta + \sin \beta \sin \theta) \quad (B-31)$$

Then

$$R_{\xi} = \sin \alpha \cos \theta \quad (B-32)$$

$$R_{h_o} = \cos \alpha \cos \beta \cos \theta + \sin \beta \sin \theta \quad (B-33)$$

$$R_{\beta} = h_o (-\cos \alpha \sin \beta \cos \theta + \cos \beta \sin \theta) \quad (B-34)$$

Also from (B-3), (B-5) and (B-6) it follows that

$$x = \xi \cos \alpha - h_o \sin \alpha \cos \beta - d \quad (B-35)$$

Then

$$x_{\xi} = \cos \alpha \quad (B-36)$$

$$x_h = -\sin \alpha \cos \beta \quad (B-37)$$

$$x_{\beta} = h \sin \alpha \sin \beta \quad (B-38)$$

To find θ_{ξ} , θ_h , and θ_{β} multiply (B-3) by \hat{e}_1 . The result is

$$\xi = (d+x) \cos \alpha + R \sin \alpha \cos \theta \quad (B-39)$$

Now multiply (B-3) by \hat{e}_3 . The result is

$$R \sin \theta = h_o \sin \beta \quad (B-40)$$

By solving (B-39) for $\cos \theta$ and (B-40) for $\sin \theta$ there is obtained

$$\tan \theta = \frac{h_o \sin \beta}{\xi \sin \alpha + h_o \cos \beta \cos \alpha} \quad (B-41)$$

Equation (B-3) is multiplied by \hat{e}_2 with the result

$$-(d+x) \sin \alpha + R \cos \alpha \cos \theta = h_o \cos \beta \quad (B-42)$$

From (B-40) and (B-42)

$$\tan \beta = \frac{R \sin \theta}{-(d+x) \sin \alpha + R \cos \alpha \cos \theta} \quad (B-43)$$

From (B-41) there is found

$$\theta_{\xi} = \frac{-h_o \sin \beta \sin \alpha}{(\xi \sin \alpha + h_o \cos \beta \cos \alpha)^2 + (h_o \sin \beta)^2} \quad (B-44)$$

$$\theta_h = \frac{\xi \sin \alpha \sin \beta}{(\xi \sin \alpha + h_o \cos \beta \cos \alpha)^2 + (h_o \sin \beta)^2} \quad (B-45)$$

$$\theta_\beta = \frac{h_o (\xi \sin \alpha \cos \beta + h_c \cos \alpha)}{(\xi \sin \alpha + h_o \cos \beta \cos \alpha)^2 + (h_o \sin \beta)^2} \quad (B-46)$$

APPENDIX C

Unlike slender-body theory the method of Reference 7 and 8 does not require the body to be slender or the angle of attack to be small. In contrast to slender-body theory, however, the contour on which the pressure is computed cannot have a blunt base because the pressure on the body near the base is affected by the shape of the wake behind it. A blunt base results in infinite velocities at the base corner and too high velocities in the entire region near the base. Consequently, the body must be extended beyond the base in a more or less realistic manner. As an approximation to the real flow the body can be extended beyond the base by use of the equation

$$\begin{aligned} \frac{R_w}{R_b} = & \left[\frac{R_w^*}{R_b} - \frac{2}{\pi} \left(\frac{dR_w^*}{dx} \right)_b N \right] \sin^2 \frac{\pi}{2} \frac{x}{NR_b} + \cos^2 \frac{\pi}{2} \frac{x}{NR_b} \\ & + \frac{2}{\pi} \left(\frac{dR_w^*}{dx} \right)_0 \sin \frac{\pi}{2} \frac{x}{NR_b} \end{aligned} \quad (C-1)$$

where $R_b = R_b(0)$ and the distance x in (C-1) is zero at the body base and positive to the rear. The non-dimensional radius R_w^* is the wake displacement radius very far behind the body. It depends on the drag coefficient of the body; the relation is

$$\frac{R_w^*}{r_m} = \sqrt{\frac{C_D}{2} [1 + (\kappa - 1) M_\infty^2]} \quad \text{(Appendix D)} \quad (C-2)$$

The arbitrariness in the choice of N and in equation (C-1) itself make it inadvisable to introduce the additional complication of $R_w^* \neq 0$ and $(\frac{dR_w^*}{dx}) \neq 0$. Consequently, instead of (C-1), there can be used

$$\frac{R_w}{R_b} = \cos^2 \frac{\pi}{2} \frac{x}{NR_b} \quad (C-3)$$

Calculations for the body of reference 17 by a program of the method of Reference 8, prepared by Hess and Clissold of Douglas Aircraft showed that $N=20$ gave the most satisfactory velocity distribution over the rear portion of the body.

When the Mach number is not zero, the pressure coefficient C_p on the body is calculated by using Gotherts Rule (page 397 of Reference 19) together with the method of Reference 8 as programmed for an electronic computer in Reference 7. The velocity distribution is calculated first for a body of thickness ratio $t \sqrt{1-M_\infty^2}$ at an angle of attack of $\alpha \sqrt{1-M_\infty^2}$. The velocities are then multiplied by $\frac{1}{1-M_\infty^2}$. This velocity distribution is used to compute the boundary layer displacement surface δ^* which is added to r_o to get R . The resultant radius R is then multiplied by $\sqrt{1-M_\infty^2}$ and the program of Reference 7 used to compute C_p at $\alpha \sqrt{1-M_\infty^2}$; this C_p is then multiplied by $\frac{1}{1-M_\infty^2}$ to get the C_p distribution used to compute the Magnus force and moment.

The force in the Y direction is given by

$$\overline{F}_Y = - \int \overline{P} \hat{n} \cdot \hat{e}_3 d\overline{A} \quad (C-4)$$

where \hat{n} is the outward unit normal to the body surface plus the displacement surface and \hat{e}_3 is a unit vector along the Y axis

Figure 6. The unit vector \hat{n} is given by

$$\hat{n} = \frac{\nabla H}{|\nabla H|} \quad (C-5)$$

where the equation of the body is

$$H(x, R, \theta) = R - R(x, \theta) = c \quad (C-6)$$

Then in the $\hat{i}, \hat{r}, \hat{\theta}$ system (Figure 8)

$$\nabla H = \hat{i} H_x + H_R \hat{r} + \frac{H_\theta}{R} \hat{\theta} \quad (C-7)$$

or

$$\nabla H = -\hat{i} R_x + \hat{r} - \frac{R_\theta}{R} \hat{\theta} \quad (C-8)$$

Then

$$\hat{n} = \frac{-\hat{i} R_x + \hat{r} - \frac{R_\theta}{R} \hat{\theta}}{\sqrt{R_x^2 + 1 + \left(\frac{R_\theta}{R}\right)^2}} \quad (C-9)$$

therefore

$$\hat{n} \cdot \hat{e}_3 = \frac{-\sin \theta + \frac{R_\theta}{R} \cos \theta}{\sqrt{R_x^2 + 1 + \left(\frac{R_\theta}{R}\right)^2}} \quad (C-10)$$

When the static pressure \bar{P} is written as

$$\bar{P} = \frac{\bar{P}_\infty}{2} \bar{V}_\infty^2 C_p + \bar{P}_\infty \quad (C-11)$$

and when it is noted that

$$\int \bar{P}_\infty \hat{n} \cdot \hat{e}_3 d\bar{A} = 0 \quad (C-12)$$

it follows that (C-4) becomes

$$\frac{\bar{F}_y}{\frac{\bar{P}_\infty}{2} \bar{V}_\infty^2 L^2} = \int_{-d}^0 \int_0^{2\pi} C_p \frac{(\sin \theta - \frac{R_\theta}{R} \cos \theta)}{\sqrt{R_x^2 + 1 + (\frac{R_\theta}{R})^2}} R \frac{\sqrt{1 + (\frac{\partial \delta^*}{\partial x} \cos \phi)^2}}{\cos \phi} d\theta dx \quad (C-13)$$

where

$$dA = R \frac{\sqrt{1 + (\frac{\partial \delta^*}{\partial x} \cos \phi)^2}}{\cos \phi} d\theta dx \quad (C-14)$$

has been used. The non-dimensional area dA is on the body plus the displacement surface.

By using $\cos \phi \frac{\partial}{\partial x} = \frac{\partial}{\partial s_x}$, $R = \frac{\partial R}{\partial s_x} s_x = s_x$ and $\cos \phi \rightarrow 0$ near the nose of a blunt body it can be shown that at the stagnation point the integrand becomes equal to $-C_{p_s} (\frac{\partial \delta^*}{\partial \theta})_s \cos \theta$. The value of $(\frac{\partial \delta^*}{\partial \theta})_s$ is zero because the δ^* surface is assumed to be continuous; therefore the integrand in (C-13) is zero at the nose of a blunt body.

The Magnus moment about the nose ($x=-d$) is given by

$$\frac{\overline{M}_y}{\frac{\rho_\infty}{2} \overline{V}_\infty^2 \overline{L}^3} = \int_{-d}^0 \int_0^{2\pi} C_p \frac{\sin \theta - \frac{R_\theta}{R} \cos \theta}{\sqrt{R_x^2 + 1 + \left(\frac{R_\theta}{R}\right)^2}} R \frac{\sqrt{1 + \left(\frac{\partial \delta^*}{\partial x} \cos \phi\right)^2}}{\cos \phi} (d+x) d\theta dx \quad (C-15)$$

where \overline{M}_y is positive as shown in Figure 1.

APPENDIX D

Far behind the body the flow is parallel to \bar{V}_∞ . The wake displacement radius \bar{R}_w^* far downstream is defined by

$$2\pi \int_{\bar{R}_w^*}^{\infty} \bar{\rho}_\infty \bar{V}_\infty \bar{y} d\bar{y} = 2\pi \int_0^{\infty} \bar{\rho} \bar{u} \bar{y} d\bar{y} \quad (D-1)$$

In (D-1) \bar{y} is measured from the wake center line and \bar{u} is parallel to \bar{V}_∞ .

Equation (D-1) can be written as

$$\int_0^{\infty} \bar{\rho}_\infty \bar{V}_\infty \bar{y} d\bar{y} - \int_0^{\bar{R}_w^*} \bar{\rho}_\infty \bar{V}_\infty \bar{y} d\bar{y} = \int_0^{\infty} \bar{\rho} \bar{u} \bar{y} d\bar{y}$$

or

$$\bar{\rho}_\infty \bar{V}_\infty \frac{\bar{R}_w^{*2}}{2} = \int_0^{\infty} (\bar{\rho}_\infty \bar{V}_\infty - \bar{\rho} \bar{u}) \bar{y} d\bar{y} \quad (D-2)$$

Then

$$\left(\frac{\bar{R}_w^*}{\bar{r}_M}\right)^2 = 2 \int_0^{\infty} (1 - \rho u) \frac{\bar{y}}{\bar{r}_M} \frac{d\bar{y}}{\bar{r}_M} \quad (D-3)$$

$$\frac{\bar{p}}{\bar{p}_\infty} = \frac{\bar{T}_\infty}{\bar{T}_s} = \frac{\bar{C}_p \bar{T}_s - \frac{\bar{V}_\infty^2}{2}}{\bar{C}_p \bar{T}_s - \frac{\bar{u}^2}{2}} \quad (D-8)$$

where in (D-8) \bar{C}_p is the specific heat at constant pressure.

Then with (D-7), (D-8) becomes

$$\frac{\bar{p}}{\bar{p}_\infty} = \frac{1}{1 - \frac{f_o \bar{V}_\infty^2}{\bar{C}_p \bar{T}_s - \bar{V}_\infty^2/2}} \quad (D-9)$$

Let

$$C_o = \frac{\bar{V}_\infty^2}{\bar{C}_p \bar{T}_s - \frac{\bar{V}_\infty^2}{2}} = \frac{\bar{V}_\infty^2}{\bar{C}_p \bar{T}_\infty} = (\kappa - 1) M_\infty^2$$

Then

$$\frac{\bar{p}}{\bar{p}_\infty} = \rho = \frac{1}{1 - f_o C_o} \quad (D-10)$$

Also with $1 - \rho u = \frac{-f_o}{1 - f_o C_o} (1 + C_o)$

and

$$\rho u (1 - u) = \frac{-f_o}{1 - f_o C_o}$$

Then (D-6) becomes

$$\frac{R_w^{*2}}{\beta_\infty} = 2 \frac{(1+C_o) \int_0^\infty \frac{f_o}{1 - f_o C_o} \frac{Y}{r_M} d\frac{Y}{r_M}}{\int_0^\infty \frac{f_o}{1 - f_o C_o} \frac{Y}{r_M} d\frac{Y}{r_M}}$$

or

$$\frac{R_w^{*2}}{\beta_\infty} = 2 [1 + (\kappa - 1) M_\infty^2] \quad (D-11)$$

or with (D-5)

$$\frac{R_w^*}{r_M} = \sqrt{\frac{C_D}{2} [1 + (\kappa - 1) M_\infty^2]} \quad (D-12)$$

TABLE I
Expressions for the $G_i(n)$

$$G_1 = \frac{n-1}{(n+1)(n+2)(2n+1)}$$

$$G_2 = \frac{1}{(n+1)(n+2)}$$

$$G_3 = \frac{n^2 + n - 1}{(n+1)(n+2)}$$

$$G_4 = \frac{n+1}{n+2}$$

$$G_5 = \frac{3n + 4}{2(n+1)(n+2)(2n+3)}$$

$$G_6 = \frac{-2(n+1)^2}{(n+2)(2n+3)}$$

$$G_7 = \frac{n}{n+1}$$

$$G_8 = \frac{1}{2n+1}$$

$$G_9 = \frac{1}{2(n+1)}$$

$$G_{10} = \frac{-n(2n+3)}{2(n+1)(n+2)}$$

$$G_{11} = \frac{-1}{(n+1)(2n+1)(2n+3)}$$

$$G_{12} = \frac{n}{(n+1)(2n+1)}$$

$$G_{13} = \frac{1}{2(n+1)(2n+1)}$$

$$G_{14} = \frac{3n}{2(n+1)(n+2)(2n+1)}$$

TABLE II
Integrals in Eq. (30) and (31)

$$\int_0^1 (u_e - u) d\zeta = J_1 \psi^2 + J_2 \psi + J_3$$

$$\int_0^1 (u_e - u) d\zeta = J_4 \psi + J_5$$

$$\int_0^1 (v_e - v) d\zeta = J_6 \psi^2 + J_7 \psi + J_8$$

$$\int_0^1 (v_e - v) d\zeta = J_{18} \psi + J_{19}$$

$$\int_0^1 (v_e^2 - v^2) d\zeta = J_9 \psi^2 + J_{10} \psi + J_{11}$$

$$\int_0^1 v(v_e - v) d\zeta = J_9 \psi^2 + J_{12} \psi + J_{13}$$

$$\int_0^1 u(v_e - v) d\zeta = J_6 \psi^2 + J_{14} \psi + J_{15}$$

$$\int_0^1 (u_e v_e - uv) d\zeta = J_6 \psi^2 + J_{16} \psi + J_{17}$$

TABLE III
Expressions for the J_i

$$J_1 = A^2 G_{11} \sin^2 a$$

$$J_2 = A [-(AG_1 - BG_2) \cos a \sin a - 2CG_5 \sin^2 a]$$

$$J_3 = A(AG_8 + B)G_7 \cos^2 a + C(-AG_3 + BG_4) \cos a \sin a + C^2 G_6 \sin^2 a$$

$$J_4 = AG_2 \sin a$$

$$J_5 = AG_7 \cos a + CG_4 \sin a$$

$$J_6 = -A^2 G_{11} \cos a \sin a$$

$$J_7 = A[(AG_{13} + BG_2) \sin^2 a + AG_{14} \cos^2 a + 2CG_5 \cos a \sin a]$$

$$J_8 = [A(AG_{12} + BG_7) - C^2 G_6] \cos a \sin a + C(AG_9 + BG_4) \sin^2 a - ACG_{10} \cos^2 a$$

$$J_9 = A^2 G_{11} \cos^2 a$$

$$J_{10} = A[AG_1 - (q_e + B)G_2] \cos a \sin a - 2ACG_5 \cos^2 a$$

$$J_{11} = A[AG_{12} + (q_e + B)G_7] \sin^2 a + C[AG_3 - (q_e + B)G_4] \cos a \sin a + C^2 G_6 \cos^2 a$$

$$J_{12} = -A[(2AG_{13} + 2G_2 B - q_e G_2) \cos a \sin a + 2CG_5 \cos^2 a]$$

$$J_{13} = [q_e(2AG_9 + B) - (A^2 G_8 + 4G_9 AB + B^2)] \sin^2 a$$

$$+ C[q_e G_4 - 2(AG_9 + BG_4)] \cos a \sin a + C^2 G_6 \cos^2 a$$

$$J_{14} = A[-(AG_{13} + BG_2) \cos^2 a + (AG_{13} + BG_2 - q_e G_2) \sin^2 a + 2CG_5 \cos a \sin a]$$

$$J_{15} = [q_e(2AG_9 + B) - (A^2 G_8 + 4ABG_9 + B^2) - C^2 G_6] \cos a \sin a$$

$$- C(AG_9 + BG_4) \cos^2 a + C(AG_9 + BG_4 - q_e G_4) \sin^2 a$$

$$J_{16} = A[-(AG_{13} + BG_2)(\cos^2 a - \sin^2 a) + 2CG_5 \sin a \cos a]$$

$$J_{17} = [q_e^2 - (A^2 G_8 + 4ABG_9 + B^2) - C^2 G_6] \cos a \sin a - C(AG_9 + BG_4)(\cos^2 a - \sin^2 a)$$

$$J_{18} = -AG_2 \cos a$$

$$J_{19} = AG_7 \sin a - CG_4 \cos a$$

TABLE IV
Program for Computation of Magnus Force and Moment
For Half-Ellipsoid

Column I are symbols that occur in program

Column II are symbols in symbol list (page vi) that correspond to symbols in Column I

<u>I</u>	<u>II</u>
A	A
AK	k
AJN	J_i
Alpha	α
B	B
C	C
COS A	$\cos \left[\alpha \frac{\pi}{180} \right]$
COSMA	$\cos a$
COSPH	$\cos \phi$
COST	$\cos \theta$
DELTA	δ
DELTAI	δ_s
DELTAZ	δ_s for $\alpha \neq 0, \omega = 0$
DELTX	Δx
DDDX	$\frac{d\delta}{dx}$
DDSTAR	
DDSTDx	$\Delta \delta^*$ $d\delta^*/dx$
DRDS	$\frac{dr_o}{ds}$
DSTARI	δ_s^*
DTHDX	$\frac{d\theta}{dx}$

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DTHOLD	$\frac{d\theta}{dx}$ at $(x-\Delta x)$
DUEXSP	$(-\frac{\partial u_e}{\partial x})_s$
EN	n
FN	G_i
G	K_1
GAMMA	η
H	K_2
P	p
PDT	$\frac{\partial \delta}{\partial \theta}$
PDX	$\frac{\partial \delta}{\partial x}$
PJNT	$\frac{\partial J_i}{\partial \theta}$
PJNX	$\frac{\partial J_i}{\partial x}$
PET	$\frac{\partial \psi}{\partial \theta}$
PPX	$\frac{\partial \psi}{\partial x}$
PRT	$\overline{\omega r_o}$
PRX	$\overline{V_\infty}$ $\frac{\partial r_o}{\partial x}$
PRX1	$\frac{1}{r_o} \frac{\partial r_o}{\partial x}$

NOLTR 72-80

PSI	ψ
PSII	ψ_s
PUET	$\frac{\partial u_e}{\partial \theta}$
PUEX	$\frac{\partial u_e}{\partial x}$
PVET	$\frac{\partial v_e}{\partial \theta}$
PVEX	$\frac{\partial v_e}{\partial x}$
QVRAT	$\sqrt{u_e^2 + v_e^2}$
QDON	$q_e \frac{\partial q_e}{\partial n}$
QDQS	$q_e \frac{\partial q_e}{\partial s}$
R	r_o
REL	Reynolds number $\frac{V_\infty \bar{L}}{V_\infty}$
RHO	I
RPDSTR	$r_c + \delta^*$
RUC	$r_o u_e \cos \phi$
SIN A	$\sin[\alpha \frac{\pi}{180}]$
SMA	a
SNSMA	Sin a
SINT	Sin θ

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T	t
THETA	θ
TWQE	$\tau_w q_e$
TWRT	$\tau_w \theta$
TWXR	$\tau_w x$
URATIO	u_e
V RATIO	v_e
XEN	Value of n for $x=x_s$
XP	Values of x chosen for printout before run on computer
XSTAG	x_s

```

PROGRAM JJTETV(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE15)
COMMON /COEF/ H,SINA,COSA,T,P,G,REL,ALPHA,XSTAG
COMMON /ITERM5/ F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12,F13,F14,
1 F15,F16,F17,F18,F19,F20,F21,F22,F23,F24,F25,F26,F27,F28,F29
COMMON /JTER45/ AJ1,AJ2,AJ3,AJ4,AJ5,AJ6,AJ7,AJ8,AJ9,AJ10,AJ11,
1 AJ12,AJ13,AJ14,AJ15,AJ16,AJ17,AJ18,AJ19,AJ20,AJ21,AJ22,AJ23,AJ24,AJ25,AJ26,AJ27,AJ28,AJ29,AJ30,AJ31,AJ32,AJ33,AJ34,AJ35,AJ36,AJ37,AJ38,AJ39,AJ40,AJ41,AJ42,AJ43,AJ44,AJ45,AJ46,AJ47,AJ48,AJ49,AJ50,AJ51,AJ52,AJ53,AJ54,AJ55,AJ56,AJ57,AJ58,AJ59,AJ60,AJ61,AJ62,AJ63,AJ64,AJ65,AJ66,AJ67,AJ68,AJ69,AJ70,AJ71,AJ72,AJ73,AJ74,AJ75,AJ76,AJ77,AJ78,AJ79,AJ80,AJ81,AJ82,AJ83,AJ84,AJ85,AJ86,AJ87,AJ88,AJ89,AJ90,AJ91,AJ92,AJ93,AJ94,AJ95,AJ96,AJ97,AJ98,AJ99,AJ100
2 AJ8X,AJ9X,AJ10X,AJ11X,AJ12X,AJ13X,AJ14X,AJ15X,AJ16X,AJ17X,AJ18X,AJ19X,AJ20X,AJ21X,AJ22X,AJ23X,AJ24X,AJ25X,AJ26X,AJ27X,AJ28X,AJ29X,AJ30X,AJ31X,AJ32X,AJ33X,AJ34X,AJ35X,AJ36X,AJ37X,AJ38X,AJ39X,AJ40X,AJ41X,AJ42X,AJ43X,AJ44X,AJ45X,AJ46X,AJ47X,AJ48X,AJ49X,AJ50X,AJ51X,AJ52X,AJ53X,AJ54X,AJ55X,AJ56X,AJ57X,AJ58X,AJ59X,AJ60X,AJ61X,AJ62X,AJ63X,AJ64X,AJ65X,AJ66X,AJ67X,AJ68X,AJ69X,AJ70X,AJ71X,AJ72X,AJ73X,AJ74X,AJ75X,AJ76X,AJ77X,AJ78X,AJ79X,AJ80X,AJ81X,AJ82X,AJ83X,AJ84X,AJ85X,AJ86X,AJ87X,AJ88X,AJ89X,AJ90X,AJ91X,AJ92X,AJ93X,AJ94X,AJ95X,AJ96X,AJ97X,AJ98X,AJ99X,AJ100X
DIMENSION PDX(50),PPX(50),PDT(50),PPT(50)
NAMLIST/INPT1/ RHO,ALPHA,T,EM,AK,REL,EN,P,ERR,XEND,MPFREQ,GAMMA
1, EZERO
WRITE(6,100)
100 FORMAT(1H1,10X14HNAMELIST INPT1 //)
READ(5,2003) NFWD
2003 FORMAT(18I4)
WRITE(6,2) NFWD
2 FORMAT(1000) FORWARDSPACE*13* FILES ON TAPE*
C DOUBLE-CHECK NUMBER OF FILES TO SKIP
JFWD=NFWD
3 READ(5,2003) NCASE
IF(NCASE.EQ.0) CALL REWIND(15)
IF(NCASE.EQ.0) STOP
NFWD=NFWD+1
IF(NFWD.NE.NCASE) STOP
IF(NCASE.EQ.1) GO TO 2004
IF(JFWD.EQ.(NFWD-1)) CALL F5FILE(15,JFWD,0)
2004 WRITE(15,2003) NCASE
READ(15,INPT1)
WRITE(6,INPT1)
IPRINT=C
MPRINT=1
CDOX=0
DDSTD=0
ERRX=2.*EZERO
ERRX2=ERRX/10.
CAMPAR=GAMMA*0.017453
MPFREQ=MPFREQ/10
PI=3.14159265358979
PI2=PI/2.
5 CALL FCALC(EN)
SINA=SIN(ALPHA*0.017453)
COSA=COS(ALPHA*0.017453)
T2=T*1
T3=SQRT(1.-T2)
10 T4=ALOG( (1.+T3)/(1.-T3) )
T5=2.*T3
G=1.-T2*(T4-T5)/(T2+T4-T5)
H=1.-T2*(T4-T5)/(T2+T4+T5*(1.-T2*2.))
G2=G*G
H2=H*H
TANAZ=(TAN(ALPHA*0.017453))**2
QC=(G2+H2*TANAZ)/(G2+H2*T2*TANAZ)

```

```

IF (ALPHA.EQ.0.) GO TO 11
IF (ALPHA.NE.0.0.AND.P.EQ.0.) GO TO 1111
PINPUT=P
AINPUT=ALPHA
UXS1=(G2*COSA-COSA-H2*SINA*SINA*(-1.))/ (T2*COSA)
UXS2=(G2+T2*H2*TANA2)**1.5/(G2+H2*TANA2)**2
DUEXSP=UXS1*UXS2
XEN=1.
XEN1=XEN+1.
XEN2=XEN+2.
EN21=2.*XEN+1.
XF10=1./EN21
XF9=XEN/XEN1
DELTAZ=SQRT(1./ (DUEXSP*REL*(2.*XF10+1.)*XF9))
ALPHA=0.
CALL QSUBR (PS,DET)
DELZP=DET
PS11=PS
P=0.
CALL QSUBR (PS,DET)
DELZ2=DET
DELTA1=DELZP+DELTAZ-DELZ2
*****
IF (NCASE.EQ.1) DELTA1=DELTA1*5.0
IF (NCASE.EQ.2) DELTA1=DELTA1*50.0
P=PINPUT
ALPHA=AINPUT
GO TO 12
1111 UXS1=(G2*COSA-COSA-H2*SINA*SINA*(-1.))/ (T2*COSA)
UXS2=(G2+T2*H2*TANA2)**1.5/(G2+H2*TANA2)**2
DUEXSP=UXS1*UXS2
XEN=1.
XEN1=XEN+1.
XEN2=XEN+2.
EN21=2.*XEN+1.
XF10=1./EN21
XF9=XEN/XEN1
DELTA1=SQRT(1./ (DUEXSP*REL*(2.*XF10+1.)*XF9))
PS11=0.
GO TO 12
11 CALL QSUBR (PS10,DELTO)
PS11=PS10
DELTA1=DELTO
12 DSTART1=DELTA1*.5
XI=-1./SQRT(1.+T2*H2*TANA2/G2)
THETA1=3.14159265358979
PS11=-1.
XSTAG=XI
XP=-.99
13 ISTEP=1
WRITE(6,1999) NCASE,ALPHA,P,REL,GAMMA,RHO
1999 FORMAT(1H1,T10*CASE*14,T25*ELLIPSOIDE,T15*ALPHA*F6.1,T35*P*F6.3,

```

```

1750*REYNOLDS NO.=E10.3.T80*GAMMA=F7.2.      199*RH0=E9.2//
1      110*X*, T22*THETA*, T37*PSI*, T50*DELTA*, T63*UR*,
1 177*VR*, T91*TRX*, T105*TRT*, T119*DSTAR*/T22*DTM/UX*,
2 T36*DELTX*, T51*SMA*,
3 T65*AA*, T79*BB*, T92*CC*, T106*PRX*, T119*COSPH*/
4 T22*RR*, T36*TWQE*, T51*QE*, T62* QDDX *, T78*QDQS*, T91*QDQN*,
5 T105*DR/DS*, T118*STHETA* /T22*RPDSTR*, T35*DDSTD*
14 X1=XI
15 X=X1
16 X2=X*X
17 XT=1.-X2*(1.-T2)
18 X21=1.-X2
19 COSPH=SQRT(X21/XT)
20 SINPH=COSPH*(-X*T/SQRT(X21))
21 R=T*SQRT(X21)
22 PRT=P*RR/T
23 TERM1=COSPH*G*COSA
24 TERM2=X/SQRT(XT)*T*H*SINA
25 PS1=PS11
26 DELTA=DELTA1
27 DSTAR=DSTAR1
28 RPDSTR=R*DSTAR
29 BETA=ATAN(PS1)
30 D2=DELTA**2
31 THETA=THETA1
32 IF(THETA.GE.0..AND.THETA.LE.PI2) STHETA=PI2-THETA
33 IF(THETA.GT.PI2..AND.THETA.LE.2*PI) STHETA=5*PI2-THETA
34 SINT=SIN(THETA)
35 COST=COS(THETA)
36 VRATIO=H*SINT*SINA
37 URATIO=TERM1-TERM2*COST
38 IF(X.NE.XSTAG) GO TO 35
39 QVRAT=0.
40 SMA=0.
41 SNSMA=0.
42 COSMA=1.
43 RJC=0.
44 DTHDX=0.
45 DRDS=-COSPH*X*T/SQRT(X21)
46 QDQS=0.
47 QDQN=0.
48 DTHOLD=DTHDX
49 DELTX=2.*RHO*COSPH*COS(GAMMA)
50 DCHECK=ABS(GAMMA)
51 IF(DCHECK.GT.90.) DELTX=1.0E-8
52 DELTAT=-2.*RHO*SIN(GAMMA)/R
53 GO TO 38
35 QVRAT=SQRT(URATIO**2+VRATIO**2)
36 IF(URATIO.NE.0.) GO TO 37
37 SMA=0.
38 SNSMA=0.
39 COSMA=1.

```

```

DTHDX=0.
RUC=0.
GO TO 38
37 SMA=ATAN(VRATIO/URATIO)
   SNSMA=VRATIO/QVRAT
   COSMA=URATIO/QVRAT
   RUC=R*URAT/O*COSPH
   DTHDX=VRATIO/RUC
38 SINAZ=SNSMA**2
   COSA2=COSMA**2
41 B=PRT*SHSMA
   A=QVRAT-B
   C=PRT*COSMA
   IF(X21.GT.0.) GO TO 42
   PUEX=0.0
   PVET=0.0
   PRX=0.
   PRX1=0.
   GO TO 43
42 PUEX=-(X**2*G*COSA/SQRT(X21))+T*H*SINA*COST)/XT**1.5
   PVET=X*T*H*SINA*SINT/SQRT(1.-X2*(1.-T2))
   PVEX=0.
   PRX=-X*T/SQRT(X21)
   PRX1=-X/X21
   IF(X.EQ.XSTAG) GO TO 43
   BRAK1=(URATIO*PUEX+VRATIO*PVEX)*COSPH
   BRAK2=(URATIO*PVET+VRATIO*PVET)/R
   UQ=URATIO/QVRAT
   VQ=VRATIO/QVRAT
   QDOS=UQ*BRAK1+VQ*BRAK2
   ODCN=-VQ*BRAK1+UQ*BRAK2
   DRDS=PRX*COSPH/SQRT(1.+R*R*DTHDX**2*COSPH**2)
43 CALL JAYS(A,B,C,COSMA,SNSMA,SINAZ,COSA2,QVRAT)
   AJ1X=AJ1
   AJ2X=AJ2
50 AJ3X=AJ3
   AJ4X=AJ4
   AJ5X=AJ5
   AJ6X=AJ6
   AJ7X=AJ7
55 AJ8X=AJ8
   AJ9X=AJ9
   AJ10X=AJ10
   AJ11X=AJ11
   AJ12X=AJ12
60 AJ13X=AJ13
   AJ14X=AJ14
   AJ15X=AJ15
   AJ16X=AJ16
   AJ17X=AJ17

```



```

65 AJ18X=AJ18
   AJ19X=AJ19
   CALL PARTLSIX,THETA,
   1PJ6T,PJ7T,PJ8T,PJ9T,PJ12T,PJ13T,PJ4X,PJ5X,PJ18T,PJ19T,
   PJ1X,PJ2X,PJ3X,PJ6X,PJ14X,PJ15X,

70 TWQE=AK*ABS(A1)**(2-EM)/(REL**EM*DELTA**EM)
   TWXR=TWQE*(COSMA-PSI*SNSMA)
   TWTR=TWQE*(SNSMA+PSI*COSMA)
   RCOS=R*COSPH
   C0=RCOS*(PSI*PSI*AJ1X+PSI*AJ2X+AJ3X)
   C1=RCOS*DELTA*(2*PSI*AJ1X+AJ2X)
75 C2=PSI*PSI*AJ6X+PSI*AJ7X+AJ8X
   C3=DELTA*(2*PSI*AJ6X+AJ7X)
   PART1=(PSI*PSI*PJ1X+PSI*PJ2X+PJ3X)*RCOS
   PART2=(PSI*PSI*AJ1X+PSI*AJ2X+AJ3X)*SINPH
   PART3=RCOS*PUEX*(PSI*AJ4X+AJ5X)
   PART4=PSI*PSI*PJ6T+PSI*PJ7T+PJ8T
80 PART5=PUET*(PSI*AJ18X+AJ19X)
   PART6=(PSI*PSI*AJ9X+PSI*AJ10X+AJ11X)*SINPH
   C4=DELTA*(PART1+PART2+PART3+PART4+PART5-PART6)
   C5=RCOS*(PSI*PSI*AJ6X+PSI*AJ14X+AJ15X)
85 C6=RCOS*DELTA*(2*PSI*AJ6X+AJ14X)
   C7=PSI*PSI*AJ9X+PSI*AJ12X+AJ13X
   C8=DELTA*(2*PSI*AJ9X+AJ12X)
   P91=(PSI*PSI*PJ6X+PSI*PJ14X+PJ15X)*RCOS
   P92=(PSI*PSI*AJ6X+PSI*AJ14X+AJ15X)*SINPH
90 P93=PSI*PSI*PJ9T+PSI*PJ12T+PJ13T
   P94=PUET*(PSI*AJ18X+AJ19X)
   P95=(PSI*PSI*AJ6X+PSI*AJ16X+AJ17X)*SINPH
   P96=RCOS*PUEX*(1*AJ4X+AJ5X)
   C9=DELTA*(P91+P92+P93+P94+P95+P96)
   DEN1=C0*C6-C1*C5
   DEN2=C2*C8-C3*C7
   CON1=R*TWXR-C4
   CON2=R*TWTR-C9
   IF(ABS(XI).LT.0.95) MPFREQ=MFREQ2
   IF(X.EQ.XSTAG) GO TO 89
   IF(X.XP192.91.92
91 IF(X.LT.-.9) XP=XP+.01
   IF(X.GT.-.9) XP=XP+.025
   IPRINT=IPRINT+1
   IF(IPRINT.LT.11) GO TO 93
   WRITE(6,1999) NCASE,ALPHA,P.REL,GAMMA,RHO
   IPRINT=1
   GOTO93

92 MPRINT=MPRINT+1
   IF(MPRINT.LT.MPFREQ) GO TO 94
   MPRINT=0

89 IPRINT=IPRINT+1
   IF(IPRINT.LT.11) GO TO 93
   WRITE(6,1999) NCASE,ALPHA,P.REL,GAMMA,RHO
   IPRINT=1

```

```

93 WRITE(6,2000) X,THETA,PSI,DELTA,URATIO,VRATIO,TWXR,TWTR,DSTAR,
1 DTHDX,DELTX,SMA,A,B,C,PRX,COSPH,R,TWGE,QVRAT,DDDX,CJQS,ODQN,DRDS
2 *STHETA,RPDSTR,DOSTDX
2000 FORMAT(1H0,1P9E14.5/15X1P8E14.5/15X1P2E14.5)
WRITE(15,2001) X,THETA,PSI,DELTA,URATIO,VRATIO,TWXR,TWTR,DSTAR
WRITE(15,2001) DTHDX,DELTX,SMA,A,B,C,PRX,COSPH
WRITE(15,2001) R,TWGE,QVRAT,DDDX,ODQS,ODQN,DRDS,STHETA
WRITE(15,2001) RPDSTR,DOSTDX
2001 FORMAT(1P9E14.5)
C VARIABLE A IN MAIN PROGRAM IS VELOCITY RATIO
94 K=1
IF(X,NE,XSIAG) GO TO 95
DELTX(K)=DELTA
PSIDX(K)=PSI
GO TO 130
95 PDX(1)=(C6*CON1-C1*CON2)/DEN1
PPX(1)=(C6*CON2-C5*CON1)/DEN1
IF(P,NE,0.) GO TO 96
PDT(1)=0.
PPT(1)=0.
GO TO 97
96 PDT(1)=(C8*(CON1-CO*PDX(1)-C1*PPX(1))-C3*(CON2-C5*PDX(1)-C6*PPX(1)
1)/DEN2
PPT(1)=(C2*(CON2-C5*PDX(1)-C6*PPX(1))-C7*(CON1-CO*PDX(1)-C1*PPX(1)
1)/DEN2
97 DELDX(1)=DELTA+(PDX(1)+PDT(1)*DTHDX)*DELTX
PSIDX(1)=PSI+(PPX(1)+PPT(1)*DTHDX)*DELTX
C*****ITERATION
102 DO 128 K=2,25
E1=C2*PDT(K-1)
E2=C3*PPT(K-1)
105 E3=C7*PDT(K-1)
E4=C8*PPT(K-1)
E5=CO*PDX(K-1)
E6=C1*PPX(K-1)
E7=C5*PDX(K-1)
110 E8=C6*PPX(K-1)
IF(P,NE,0.) GO TO 111
PDT(K)=0.
PPT(K)=0.
GO TO 112
111 PDT(K)=(C8*(CON1-E5-E6)-C3*(CON2-E7-E8))/DEN2
PPT(K)=(C2*(CON2-E7-E8)-C7*(CON1-E5-E6))/DEN2
112 PDX(K)=(C6*(CON1-E1-E2)-C1*(CON2-E3-E4))/DEN1
PPX(K)=(CO*(CON2-E3-E4)-C5*(CON1-E1-E2))/DEN1
DDDX=PDX(K)+PDT(K)*DTHDX
DELDX(K)=DELTA+DDDX*DELTX
PSIDX(K)=PSI+(PPX(K)+PPT(K)*DTHDX)*DELTX
115 TEST1=(DELDX(K)-DELDX(K-1))/DELDX(K-1)
IF(ABS(TEST1),LE,ERR) GO TO 118
GO TO 128

```

```

118 IF(P.EQ.0.) GO TO 130
    TEST2=(PSIDX(K)-PSIDX(K-1))/PSIDX(K-1)
    IF(ABS(TEST2).LE.ERR) GO TO 130
128 CONTINUE
500 WRITE(6,500)
    FORMAT(1H1,10X31HNO CONVERGENCE IN,25 ITERATIONS )
    STOP
130 GO TO (135,140),ISTEP
135 THETA1=THETA+DELTAT
    DELTA1=DELTA+DELTA1
    PS11=PSIDX(K)
    DSTAR1=DSTAR+.5
    XI=X+DELTA
    ISTEP=2
    GO TO 14
140 IF(ALPHA.EQ.0.)GOTO144
    IF(THETA.EQ.0.)GOTO144
    IF(THETA.EQ.PI)GOTO144
    DXTERM=ARS(1.-DTHOLD/DTHDX)
    IF(DXTERM.GE.ERRX2.AND.DXTERM.LE.ERRX) GO TO 144
    IF(DXTERM.LT.ERRX2) DELTX=2.*DELTA
    IF(DXTERM.GT.ERRX) DELTX=0.5*DELTA
144 IF(DELTX.GT.0.0001) DELTX=0.0001
    DELTA1=DTHDX*DELTA
    THETA1=THETA+DELTA
    IF(URATIO.NE.0.) GO TO 145
    DDSTAR=-DDSTAR*DELTA*PRX1
    GO TO 150
145 DD51=-(PUEX/URATIO+PRX1+PVET/RUC)*DSTAR*DELTA
    DD52=DELTA*DELTA/URATIO*(PJ4X*PSI+AJ4X*PPX(K)+PJ5X)
    DD53=DELTA/URATIO*(PDX(K)+DELTA/R*PRX1*(PSI*AJ4X+AJ5X)
    DD54=DELTA*DELTA/RUC*(AJ18X*PPT(K)+PJ18*PSI+PJ19T)
    DD55=DELTA*PPT(K)/RUC*(AJ18X*PSI+AJ19X)
    DDSTAR=DD51+DD52+DD53+DD54+DD55
150 PSIX=PSI+(PPX(K)+PPT(K)*DTHDX)*DELTA
    *****
    XLIM=-.9980
    IF(NCASE.EQ.2) XLIM=-.950
    IF(X.GT.XLIM) GO TO 174
    DELTA1=DELTA
    DSTAR1=DSTAR
    PS11=PSI
    GO TO 175
174 IF(DDOX.LT.0.) DDDX=0.
    DELTA1=DELTA+DDOX*DELTA
175 PS11=PSIX
    DELTA1=DELTA
    DSTAR1=DSTAR+DDSTAR
176 THETA1=THETA
177 XI=X+DELTA
    IF(ABS(X-XP).GT.DELTA) GO TO 178
    DELTA1=ABS(X-XP)

```

```

XI=XP
DELTA1=DELTA+DDX*DELTX
PSI1=PSI+(PPX(K)+PPT(K)*DTHDX)*DELTX
THETA1=THETA+DTHDX*DELTX
IF(URATIO.NE.0.) GO TO 1450
DDSTAR=-DSTAR*DELTX*PRX1
GO TO 1500
1450 DDS1=-((PUEX/URATIO+PRX1+PVEI/RUC)*DSTAR*DELTX
      DDS2=DELTA*DELTX/URATIO*(PJ4X*PSI+AJ4X*PPX(K)+PJ5X)*
      DDS3=DELTX/URATIO*(PDX(K)+DELTA/R*PRX1)*(OSI*AJAX+AJ5X)
      DDS4=DELTA*DELTX/RUC*(AJ18X*PPT(K)+PJ18T*PSI+PJ19T)
      DDS5=DELTX*PDT(K)/RUC*(AJ18X*PSI+AJ19X)
      DDSTAR=DDS1+DDS2+DDS3+DDS4+DDS5
      DDSTDX=DDSTAR/DELTX
1500 DSTAR1=DSTAR+DDSTAR
178 IF(XI.GT.XEND) ENDFILE 15
    IF(XI.GT.XEND) GO TO 3
    DTHOLD=DTHDX
    GO TO 14
    END

```

RETURN
END

```

SUBROUTINE FCALC (EN)
COMMON /FTEMS/ F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12,F13,F14,
1 F15,F16,F17,F18,F19,F20,F21,F22,F23,F24,F25,F26,F27,F28,F29
1 EN1=EN+1,
EN2=EN+2,
EN21=2.*EN+1,
F1=(EN-1.)/(EN1*EN2* EN21)
5 F2=1./ (EN1*EN2)
F3=(EN+2+EN-1.)/(EN1*EN2)
F4=EN1/EN2
F5=3./(2.*EN2*EN21)
F6=(2.*EN+3.)/(2.*EN2)
10 F7=(3.*EN+4.)/(2.*EN1*EN2*(2.*EN+3.))
F8=(-2.*EN1*2)/(EN2*(2.*EN+3.))
F9=EN/EN1
F10=1./EN21
F11=1./ (2.*EN1)
15 F12=1./ (2.*EN2)
F13=EN1/( (2.*EN+3.)*EN2)
F14=-EN*(2.*EN+3.)/(2.*EN1*EN2)
F15=-F3
F16=F1
20 F17=-F2
F18=-1./ (EN1*EN21*(2.*EN+3.))
F19=-F7
F20=-2.*F7
F21=EN/ (EN1*EN21)
25 F22=1./ (2.*EN1*EN21)
F23=3.*EN/(2.*EN1*EN2*EN21)
F24=-F16
F25=F23-2.*F7
F26=F21-F9
30 F27=F3+F4
F28=F26
WRITE(6,40) F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12,F13,F14,F15,
1 F16,F17,F18,F19,F20,F21,F22,F23,F24,F25,F26,F27,F28
40 FORMAT(1H0,10X10HTEMS .../ (10X1P8E15,5))

RETURN
END

```

```

SUBROUTINE PARTLS(X,THETA,
1 PJ1X,PJ6T,PJ7T,PJ8T,PJ9T,PJ12T,PJ13T,PJ4X,PJ5X, PJ18T,PJ19T,
COMMON /COEF/ H,SINA,COSA,T,P,G,REL,ALPHA,XSTAG
COMMON /JTERM5/ AJ1,AJ2,AJ3,AJ4,AJ5,AJ6,AJ7,AJ8,AJ9,AJ10,AJ11,
1 AJ12,AJ13,AJ14,AJ15,AJ16,AJ17,AJ18,AJ19,AJ2X,AJ3X,AJ6X,AJ7X,
2 AJ8X,AJ9X,AJ12X,AJ13X,AJ14X,AJ15X,AJ4X,AJ5X,AJ18X,AJ19X
DIMENSION Z1(2,2), Z2(2,2), Z3(2,2), Z6(2,2), Z7(2,2), Z8(2,2),
1 Z9(2,2), Z12(2,2), Z13(2,2), Z14(2,2), Z15(2,2),
2 Z4(2,2), Z5(2,2), Z18(2,2), Z19(2,2)

DELTX=0.0001
DELTAT=0.0001
1 T2=T+T
XX=X
TK=THETA
NT=2
5 XKEEP=XK-DELTX
DO 61 I=1,2
XKFFP=XKFFP+DELTX
X2=XKEEP**2
XT=1.-X2*(1.-T2)
10 X21=1.-X2
COSPH=SQRT(X21/XT)
SINPH=COSPH*(-X*T/SQRT(X21))
R=T*SQRT(X21)
PRT=P*R/T
15 TERM1=COSPH*G+COSA
TERM2=XKEEP/SQRT(XT)*T*H*SINA
TK=TK-DELTAT
DO 58 J=1,NT
IF(1.EQ.1.AND.J.EQ.1) GO TO 46
20 TK=TK+DELTAT
SINT=SIN(TK)
COST=COS(TK)
VR=-H*SINT*SINA
UR=TERM1-TERM2*COST
IF(XKEEP.NE.XSTAG) GO TO 30
OVRAT=0.
SMA=0.
SNSMA=0.
COSMA=1.
GO TO 33
30 OVRAT=SQRT(UR**2+VR**2)
IF(UR.NE.0.) GO TO 52
SMA=0.
SNSMA=0.
COSMA=1.
GO TO 33
32 SMA=ATAN(VR/UR)
SNSMA=VR/OVRAT
COSMA=UR/OVRAT
33 SINA2=SNSMA**2

```

Z9(I,J)=AJ9
Z12(I,J)=AJ12
Z13(I,J)=AJ13
Z14(I,J)=AJ14
Z15(I,J)=AJ15
Z18(I,J)=AJ18
Z19(I,J)=AJ19

GO TO 58

46 Z1(I,J)=AJ1X
Z2(I,J)=AJ2X
Z3(I,J)=AJ3X
Z4(I,J)=AJ4X
Z5(I,J)=AJ5X
Z6(I,J)=AJ6X
Z7(I,J)=AJ7X
Z8(I,J)=AJ8X
Z9(I,J)=AJ9X
Z12(I,J)=AJ12X
Z13(I,J)=AJ13X
Z14(I,J)=AJ14X
Z15(I,J)=AJ15X
Z18(I,J)=AJ18X
Z19(I,J)=AJ19X

50

TK=THETA
IF(ALPHA.EQ.0.) GO TO 59

58 CONTINUE

59 NT=1

TK=THETA

61 CONTINUE

C PARTIALS WITH RESPECT TO X

PJ1X=(Z1(2,1)-Z1(1,1))/DELTX
PJ2X=(Z2(2,1)-Z2(1,1))/DELTX
PJ3X=(Z3(2,1)-Z3(1,1))/DELTX
PJ4X=(Z4(2,1)-Z4(1,1))/DELTX
PJ5X=(Z5(2,1)-Z5(1,1))/DELTX
PJ6X=(Z6(2,1)-Z6(1,1))/DELTX
PJ14X=(Z14(2,1)-Z14(1,1))/DELTX
PJ15X=(Z15(2,1)-Z15(1,1))/DELTX

65

C PARTIALS WITH RESPECT TO THETA

1=ALPHA*NE.0.1 GO TO 68
PJ6T=0.0
PJ7T=0.0
PJ8T=0.0
PJ9T=0.0
PJ12T=0.0
PJ13T=0.0
PJ18T=0.0


```

PJ1QT=0.
GO TO 72
68 PJ6T=(Z6(1,2)-Z6(1,1))/DELTAT
PJ7T=(Z7(1,2)-Z7(1,1))/DELTAT
70 PJ8T=(Z8(1,2)-Z8(1,1))/DELTAT
COSA2=COSMA**2
B=PT*SNMA
A=QVRAT-B
C=PT*COSMA
CALL JAYS(A,B,C,COSMA,SNMA,SINA2,COSA2,QVRAT)
Z1(1,J)=AJ1
Z2(1,J)=AJ2
Z3(1,J)=AJ3
Z4(1,J)=AJ4
Z5(1,J)=AJ5
Z6(1,J)=AJ6
Z7(1,J)=AJ7
40 Z8(1,J)=AJ8
PJ9T=(Z9(1,2)-Z9(1,1))/DELTAT
PJ12T=(Z12(1,2)-Z12(1,1))/DELTAT
PJ13T=(Z13(1,2)-Z13(1,1))/DELTAT
PJ18T=(Z18(1,2)-Z18(1,1))/DELTAT
PJ19T=(Z19(1,2)-Z19(1,1))/DELTAT
72 RETURN
END

```

```

SUBROUTINE QSUBR(P$II,DELTAI)
COMMON /CUEF/ H,SIN(A,COSA,T,P,G,REL,ALPHA
DIMENSION Q(25),E(25)
EN=1.
EN1=FH+1.
EN2=EN+2.
EN21=2.*EN+1.
F7=(3.*EN+4.)/(2.*EN1*EN2*(2.*EN+3.))
F8=(-2.*EN1*#2)/(EN2*(2.*EN+3.))
F9=EN/EN1
F10=1./EN21
F11=1./12.*EN1)
F18=-1./1*(EN1*EN21*(2.*EN+3.))
F22=1./12.*EN1*EN21)
C1=(3.*F10+1.)*F9
C2=((P*Y)/G)*#2
C3=16.*F18*F11*#2
Q(1)=1.0
Q(2)=0.*95
DO 20 I=1,25
E(I)=Q(I)-C1+C2*(C3/(Q(I)+4.*F22)*#2+(8.*F11*F7)/(Q(I)+4.*F22)
1 +F8)
IF(ABS(E(I)).LE.0.00001) GO TO 25
IF(I-LT,2) GO TO 20
Q(I+1)=Q(I)-E(I)*(Q(I)-Q(I-1))/(E(I)-E(I-1))
20 CONTINUE
STOP
PSII=(-4.*F11*(P*Y)/G)/(Q(I)+4.*F22)
DELTAI=(T*#2)/(Q(I)+1.*G*REL)
DELTAI=SQRT(DELTAI)
WRITE(6,10) PSII,DELTAI,Q(I)
10 FORMAT(10,11HQSUB--PSII=1PE13.5,5X7HDELTAI=1PE13.5,5X2H0=1PE13.5)
RETURN
END

```



```

C      REWIND IUNIT
      IUTAB(IUNIT) = 1
      RETURN

C      SKIP FILES FORWARD
      ENTRY FSFILE
      N=1
      GO TO 1000
C      BACKSPACE FILE
      ENTRY RSFILE
      N=2
      GO TO 1000
C      REWIND FILE
      ENTRY RWFILE
      N=3
      GO TO 1000
C      TEST FOR EOF
      ENTRY ENDFIL
      GO TO 5000
C      1000 IF(IFLAG.NE.0) GO TO 1999
      IFLAG = 1
      ISTAT = 0
      ICUFIL = 1
      1999 ITEST = LOCF(IP1)
      IF(ITEST.EQ. 0) GO TO 9999
      IUNIT = IP1
      IF(IUNIT.LE. 0 .OR. IUNIT.GT. 99) GO TO 9999
      ICUFIL = IUTAB(IUNIT)
      ITEST = LOCF(IP2)
      IF(ITEST.EQ. 0) GO TO 1001
      NFILES = IP2
      GO TO 1002
      1001 NFILES = 1
      1002 ITEST = LOCF(IP3)
      IF(ITEST.EQ. 0) GO TO 1003
      MODE = IP3
      GO TO 1004
C      ASSUME BCD IF MODE NOT SPECIFIED
      1003 MODE = 0
C      1004 IF(N.EQ. 3) MODE = NFILES
      GO TO (2000,3000,4000) ,N
C
C      FORWARD SKIP NFILES
      2000 IF (MODE.NE. 0) GO TO 2500
      2001 READ(IUNIT,6000)
      6000 FORMAT(13A10,A6)

```

```

2002 IF(EOF(IUNIT)) 2002,2001,2002
      ICUFIL= ICUFIL+1
      NFILES = NFILES-1
      IF(NFILES.LE.0) GO TO 7777
      GO TO 2001
      BINARY MODE
C
C
C
C
2500 READ(IUNIT)
      IF(EOF(IUNIT)) 2502,2500,2502
2502 ICUFIL = ICUFIL+1
      NFILES = NFILES-1
      IF(NFILES.LE.0) GO TO 7777
      GO TO 2500
C
C      BACK SPACE FILES
C
3000 IF(NFILES.LE.0) GO TO 9999
3010 CONTINUE
      REWIND IUNIT
      NSKIP = ICUFIL-NFILES-1
      ICUFIL = NSKIP+1
      IF(NSKIP .LE. 0) GO TO 7777
      IF(MODE.NE.0) GO TO 3500
3001 READ(IUNIT,6000)
      IF(EOF(IUNIT)) 3100,3001,3100
3100 CONTINUE
      NSKIP = NSKIP-1
      IF(NSKIP.LE.0) GO TO 7777
      GO TO 3001
C
C      BINARY MODE
C
3500 DO 3502 I= 1,NSKIP
3501 READ(IUNIT)
      IF(EOF(IUNIT)) 3502,3501,3502
3502 CONTINUE
      GO TO 7777
C
C      END FILE TEST
C
5000 CONTINUE
      ITEMP = LOCF(IP1)
      IF(ITEMP.EQ. 0) GO TO 9999
      IUNIT = IP1
      ICUFIL = IUTAB(IUNIT)
      IF(EOF(IUNIT)) 5001,500,5001
5001 ICUFIL = ICUFIL+1
      IP2 = 1
      GO TO 7777
5050 IP2 =2

```

```

C      GO TO 7777
C      REWIND CURRENT FILE
C      4000 NFILES = 0
C      GO TO 3010
C      COMMON RETURN POINT
C      RESET FILE IN PROPER SLOT
C      7777 IF(1CUFIL .LE. 0) 1CUFIL = 1
C      1UTAB(1UNIT) = 1CUFIL
C      RETURN
C      ERROR MESSAGE
C      9999 PRINT 9000
C      9000 FORMAT(//'.* $$$ ILLEGAL OR NO PARAMETER IN CALL TO FSB$FL $$$*.'//
C      X/)
C      CALL EXIT
C      END

```

```

PROGRAM VNPRSPL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE15,
1 TAPE16)
DIMENSION XPICK(50),JORDER(50),AA(300,27),ZTHETA(45,35),ZX(45,35),
1 ZDTHDX(45,35),ZDSTAR(45,35),ZRDPSTR(45,35),XSPLIN(181),
1YSPL1(181),YSPL2(181),YSPL3(181),YSPL4(181),YSPL5(181),C1(4,40),
1C2(4,40),C3(4,40),C4(4,40),C5(4,40),XIN(181),ZDDSTUX(45,35),
1ZPRX(45,35)
WRITE(6,302)
302 FORMAT(1H1*          VALUES READ FROM TAPE*,I7*CASE*,I21*X*,I25*ORDER*,
1T39*THETA*,I53*STAR*,I69*PRX*,I81*DTHDX*,I94*RPDSTR*,I108*DDSTDX*,
1/)
C JPICKS=NUMBER OF X VALUES TO BE PROCESSED
READ(5,301) JPICKS
READ(5,501) (XPICK(J),J=1,JPICKS)
501 FORMAT(12F6,3)
NOMIT=0
PI=3.14159265358979
TVOP1=2.*PI
NCASES=TOTAL NUMBER OF CASES ON TAPE
C JORDER(N) IS SEQUENCING OF CASES IN ASCENDING VALUES OF GAMMA FOR
C ASCENDING VALUES OF THETA
C JORDER(N) INPUT AS 0 IF CASE IS NOT TO BE PROCESSED
READ(5,301) NCASES
READ(5,301) (JORDER(N),N=1,NCASES)
301 FORMAT(24I3)
CALL F$FILE('16,27,0')
1 READ(15,9) NCASE
N=0
9 FORMAT(14I4)
12 N=N+1
IF(N.GT.300) STOP4
READ(15,20)(AA(N,I),I=1,9)
20 FORMAT(9E14,5)
CALL ENDFIL(15,ISTAT)
ISTAT=1 IF END OF FILE.      ISTAT=2 IF NO END OF FILE
IF(ISTAT.EQ.1) GO TO 30
25 READ(15,21)(AA(N,I),I=10,27)
21 FORMAT(8E14,5)
GO TO 12
30 NH=JORDER(NCASEF)
IF(NH)2,22,23
22 NOMIT=NOMIT+1
IF(NCASE.EQ.NCASES) GO TO 37
GO TO 1
23 NLINES=N-1
DO 35 J=1,JPICKS
DO 40 N=1,NLINES
*** XDIF CHECK AT VALUE OF X=0.0
XDIF=ABS(AA(N,1)-XPICK(J))
IF(XDIFF.GE.1.E-11) GO TO 40
ZTHETA(J,NH)=AA(N,2)
ZPRX(J,NH)=AA(N,16)

```

```

ZDDSTDX(J,NN)=AA(N,27)
ZDTHDX(J,NN)=AA(N,10)
ZDSTAR(J,NN)=AA(N,9)
ZRPOSTR(J,NN)=AA(N,26)
ZX(J,NN)=AA(N,1)
WRITE(6,303) NCASE,ZX(J,NN),NN,ZTHETA(J,NN),ZDSTAR(J,NN),
12PRX(J,NN),ZDTHDX(J,NN),ZRPOSTR(J,NN),ZDDSTDX(J,NN)
303 FORMAT(110,1P6E14.5,15,6E14.5)
GO TO 35
40 CONTINUE
STOP
35 CONTINUE
IF(NCASE.LT.NCASES) GO TO 1
37 NPTS=NCASES-NOMIT
DO 70 J=1,JPIKES
DO 311 I=1,3
II=NPTS-(3-I)
XSPLIN(I)=-(TWOPI-ZTHETA(J,II))
YSPL1(I)=ZDSTAR(J,II)
YSPL2(I)=ZPRX(J,II)
YSPL3(I)=ZDTHDX(J,II)
YSPL4(I)=ZRPOSTR(J,II)
YSPL5(I)=ZDDSTDX(J,II)
XSPLIN(NPTS+3+I)=TWOPI+ZTHETA(J,I)
YSPL1(NPTS+3+I)=ZDSTAR(J,I)
YSPL2(NPTS+3+I)=ZPRX(J,I)
YSPL3(NPTS+3+I)=ZDTHDX(J,I)
YSPL4(NPTS+3+I)=ZRPOSTR(J,I)
YSPL5(NPTS+3+I)=ZDDSTDX(J,I)
311 CONTINUE
DO 312 NN=1,NPTS
XSPLIN(NN+3)=ZTHETA(J,NN)
YSPL1(NN+3)=ZDSTAR(J,NN)
YSPL2(NN+3)=ZPRX(J,NN)
YSPL3(NN+3)=ZDTHDX(J,NN)
YSPL4(NN+3)=ZRPOSTR(J,NN)
YSPL5(NN+3)=ZDDSTDX(J,NN)
MPAIR=NPTS+6
WRITE(6,310)XPICK(J),(NN,XSPLIN(NN),YSPL1(NN),YSPL2(NN),YSPL3(NN),
1YSPL4(NN),YSPL5(NN),NN=1,MPAIR)
310 FORMAT(1H1,* ORDERED VALUES GOING INTO SPLINE FIT. X=F7.4/T7,
1*NN,118*THETA,132*ZDSTAR,148*PRX,160*ZDTHDX,173*ZRPOSTR,187,
1*DCSTDX,178,1P6E14.5)
CALL SPLICON(XSPLIN,YSPL1,MPAIR,C1)
CALL SPLICON(XSPLIN,YSPL2,MPAIR,C2)
CALL SPLICON(XSPLIN,YSPL3,MPAIR,C3)
CALL SPLICON(XSPLIN,YSPL4,MPAIR,C4)
CALL SPLICON(XSPLIN,YSPL5,MPAIR,C5)
XINCR=2.*PI/180.
XIN(1)=0.
DO 314 NN=2,181

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```

314 XIN(NN)=XIN(NN-1)+XINCR
    WRITE(6,200)
200 FORMAT(1H1,I8*ALPHA*,I25*X*,I37*RPDSTR*,I53*THETA*,I68*DDSTAR*,I62*
1*DDSTD*,I97*PDSTD*,I7*DDSTART*,I24*PRX*,I39*CSE*,I54*BTA*,I70*
1*RO*,I82*DRODST*,I97*DRODCS*/I9*VNPR*,I23*GAMA*,I39*PR*)
    IPRINT=0
    DO 60 NN=1,I31
        THETA=XIN(NN)
        CALL SPLINE(XSPLIN,YSPL1,MPAIR,C1,THETA,DDSTAR,DERIV,IOK)
        IF(IOK.EQ.0) WRITE(6,320) THETA
320 FORMAT(///** OUT OF RANGE FOR SPLINE FIT*/THETA=E14.5)
        IF(IOK.EQ.0) STOP3
        CALL SPLINE(XSPLIN,YSPL2,MPAIR,C2,THETA,PRX,DUM,IOK)
        IF(IOK.EQ.0) STOP3
        CALL SPLINE(XSPLIN,YSPL3,MPAIR,C3,THETA,OTHDX,DUM,IOK)
        IF(IOK.EQ.0) STOP3
        CALL SPLINE(XSPLIN,YSPL4,MPAIR,C4,THETA,RPDSTR,DUM,IOK)
        IF(IOK.EQ.0) STOP3
        CALL SPLINE(XSPLIN,YSPL5,MPAIR,C5,THETA,DDSTD*,DUM,IOK)
        IF(IOK.EQ.0) STOP3
        X=XPICK(J)
        DDSTAR=DERIV
        ALPHA=4.
        A=1.
        T=.1
        ALPHA=ALPHA*PI/180.
        FR=1.
        R=T *SORT(1,-X*X)
        PDSTD=DDSTD*-OTHDX*DDSTAR
        FX=-PRX-PDSTD
        FHTA=-A*DDSTAR
        DRDCE=SIN(ALPHA)*COS(THETA)
        Z1=(R+DDSTAR)*SIN(THETA)/(-1.+X)*SIN(ALPHA)+(R+DDSTAR)*
        ICOS(ALPHA)*COS(THETA)
C 21 IS NOT USED
        XK1=-(A+X)*SIN(ALPHA)+(R+DDSTAR)*COS(ALPHA)*COS(THETA)
        XK2=(R+DDSTAR)*SIN(THETA)
        XK2K1=ABS(XK2/XK1)
        BTA=ATAN(XK2K1)
50 FORMAT( 5X*NUMERATOR XK2 =E13.5/5X*DENOMINATOR XK1 =E13.5/
15X,*ATAN(XK2/XK1)=BTA1 =E13.5)
        CSINAL=SIN(ALPHA)
        CCOSAL=COS(ALPHA)
        CSINTH=SIN(THETA)
        CCOSTH=COS(THETA)
401 FORMAT(5X*SIN(ALPHA)=E13.5/5X*COS(ALPHA)=E13.5/5X*SIN(THETA)=*
1E13.5/5X*COS(THETA)=E13.5)
        IF(XK2.GT.0.0.AND.XK1.GT.0.0) BTA=BTA1
        IF(XK2.GT.0.0.AND.XK1.LT.0.0) BTA=PI-BTA1
        IF(XK2.LT.0.0.AND.XK1.LT.0.0) BTA=PI+BTA1
        IF(XK2.LT.0.0.AND.XK1.GT.0.0) BTA=2*PI-BTA1
        IF(THETA.EQ.0.0) BTA=0.

```

```

IF (THETA.EQ.PI) BTA=PI
RO=-(1.+X)*SIN(ALPH)*COS(BTA)+(R+DS*AR)*(COS(ALPH)*COS(BTA)
1*COS(THETA)+SIN(BTA)*SIN(THETA))
DROU=A*RO*(-COS(ALPH)*SIN(BTA)*COS(THETA)+COS(BTA)*SIN(THETA))
DRDRO=COS(ALPH)*COS(BTA)*CUS(THETA)+SIN(BTA)*SIN(THETA)
DXDCSE=COS(ALPH)
CSE=(1.+X)*COS(ALPH)+(R+DS*AR)*SIN(ALPH)*COS(THETA)
DXDBTA=A*RO*SIN(ALPH)*SIN(BTA)
DXDRO=-SIN(ALPH)*COS(BTA)
DEN=(CSE*SIN(ALPH)+RO*COS(BTA)*COS(ALPH))*2+(RO*SIN(BTA))*2
DTDCSE=-(RO*SIN(BTA)*SIN(ALPH))/(DEN*A)
DTHDD=(RO*CSE*SIN(ALPH)*COS(BTA)+RO*RO*COS(ALPH))/DEN
DTHURO=(CSE*SIN(ALPH)*SIN(BTA))/(DEN*A)
DEN1=FR*DRDRO+FX*DXDRO+FTHTA*DTHTDRO
DRODCS=-(FR*DRDRO+FX*DXDCSE+FTHTA*DTHTDRO)/(DEN1*A)
DRODBT=-(FR*DRDRO+FX*DXDBTA+FTHTA*DTHTDRO)/(DEN1*A)
405 FORMAT(5X,DEN=E13.5)
404 FORMAT(15*DRDRO=E13.5/15*DXDRO=E13.5/15*DTHTDRO=E13.5)
402 FORMAT(1/15*DRDCSE=E13.5/15*FX=E13.5/15*DXDCSE=E13.5/15*FTHTA=,
1E13.5/15*DTDCSE=E13.5/15*DEN1=E13.5)
403 FORMAT(1/15*DRDRO=E13.5/15*DXDBTA=E13.5/15*DTHTDRO=E13.5)
VNPB=DRODCS/SQRT(1.+{(DRODBT/RO)**2)
242 FORMAT(10 RO=E1PE13.5, X=E1PE13.5, SIN(ALPH)=E1PE13.5, bT
1A=E1PE13.5)
CHEC4=RO**2+2.*RO*(1.+X)*COS(BTA)*SIN(ALPH)+(1.+X)*SIN(ALPH)**2
IF (CHEC4.LE.0.) WRITE(6,420) CHEC4
IF (CHEC4.LE.0.) STOP6
420 FORMAT(//, CHEC4=E1PE15.5)
RPR=SQRT(RO**2+2.*RO*(1.+X)*COS(BTA)*SIN(ALPH)
1+(1.+X)*SIN(ALPH)**2)
IF (RPR.LE.1.0E-6) WRITE(6,421) RPR
IF (RPR.LE.1.0E-6) STOP6
421 FORMAT(//, RPR=E1PE15.5)
406 FORMAT(10 RPR=E1PE13.5, X=E1PE13.5, RO=E1PE13.5, X=E1PE13.5)
PIHALF=PI/2.
TWOP1=2.*PI
PI15=1.5*PI
IF (BTA.GE.0. AND BTA.LE.PIHALF) GO TO 600
IF (BTA.GT.PI15 AND BTA.LT.TWOP1) GO TO 605
IF (BTA.EQ.PI) GAMA=PI15
IF (BTA.EQ.TWOP1) GAMA=PIHALF
IF (BTA.EQ.PI) GO TO 450
IF (BTA.EQ.TWOP1) GO TO 450
TARG=(1.0+X)*SIN(ALPH)/RO
IF (ABS(TARG).GT.1.0) WRITE(6,445)
445 FORMAT(51H CALCULATION OF TERM = ASIN((1.0+X)*SIN(ALPH)/RO))
IF (ABS(TARG).GT.1.0) CALL CHECS(TARG)
TERM=ASIN(TARG)
IF (TARG.EQ.1.0) WRITE(6,446) TERM
446 FORMAT(10 TERM=E1PE13.5)
IF (BTA.GT.PIHALF AND BTA.LT.PIHALF+TERM) GO TO 601
IF (BTA.GT.PIHALF+TERM AND BTA.LT.PI) GO TO 602

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```

IF(BTA.GT.PI.AND.BTA.LI.(PI15-TERM)) GO TO 603
IF(BTA.GT.(PI15-TERM).AND.UTA.LE.PI15) GO TO 604
IF(BTA.EQ.(PIHALF+TERM)) GAMA=PI
IF(BTA.EQ.(PI15-TERM)) GAMA=TWOPI
IF(BTA.EQ.(PIHALF+TERM)) GO TO 450
IF(BTA.EQ.(PI15-TERM)) GO TO 450
WRITE(6,447) BTA,TERM
447 FORMAT(/ * BTA =1PE13.5,4X*TERM =1PE13.5/* NO CONDITIONS SATI
1SFIED FOR GAMA---STOP,*)
STOP
600 ARG=RO*SIN(BTA)/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=PIHALF+ASIN(ARG)
GO TO 450
601 ARG=RO*COS(BTA-PIHALF)/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=PIHALF+ASIN(ARG)
GO TO 450
602 ARG=(RO*SIN(BTA-PIHALF)-(1.0+X)*SIN(ALPH))/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=PI+ASIN(ARG)
GO TO 450
603 ARG=(RO*SIN(PI15-BTA)-(1.0+X)*SIN(ALPH))/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=TWOPI-ASIN(ARG)
GO TO 450
604 ARG=RO*COS(PI15-BTA)/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=PIHALF-ASIN(ARG)
GO TO 450
605 ARG=(RO*SIN(BTA-PI15)+(1.0+X)*SIN(ALPH))/RPR
IF(ABS(ARG).GT.1.0) CALL CHECS(ARG)
GAMA=ASIN(ARG)
PRINT HEADER AND 14 BLOCKS OF DATA ON EACH PAGE
C 450 IPRINT=IPRINT+1
IF(IPRINT.GT.14) WRITE(6,200)
IF(IPRINT.GT.14) IPRINT=1
WRITE(6,201)ALPHA,X,RPDSTR,THE.3.0,DSIAR,DDSDTX,PDSDTX,DSIART,PRX,
1CSE,BTA,RO,DRODST,DRODCS,VNPR,GAMA,4,1
201 FORMAT(1P7E15.5)
WRITE(16,2) ALPHA,X,RPDSTR,THE.3.0,DSIAR,DDSDTX,PDSDTX
WRITE(16,2) DSIART,PRX,CSE,BTA,RO,DRODST,DRODCS
WRITE(16,2) VNPR,GAMA,RPR
2 FORMAT(1P7E15.5)
60 CONTINUE
ENDFILE'16.
70 CONTINUE
STOP
END

```

NOLTR 72-80

```
SUBROUTINE CHECS(ARG)
DIFF=ABS(ARG)-1.0
WRITE(6,10) ARG,DIFF
10 FORMAT(/# ABSOLUTE VALUE OF ARGUMENT FOR ASIN IS GREATER THAN 1.
10 FOR COMPUTATION OF GAMMA,*/# ARG=1PE17,10.6X*(ABS(ARG)-1.0)=*
16PE22,10)
IF(DIFF.LT.1.0E-5) ARG=1.0
IF(DIFF.LT.1.0E-5) RETURN
STOP10
END
```

```

SUBROUTINE SPLICON(X,Y,M,C)
  INPUT X AND Y TABLE ARRAYS IN ASCENDING VALUES OF X
  INPUT M = NUMBER OF (X,Y) PAIRS IN TABLE
  OUTPUT C ARRAY OF CONSTANTS
  DIMENSION X(N),Y(N),D(N),P(N),E(N),C(4,N),A(N,3),B(N),Z(N) WHERE N=GE.M
  DIMENSION X(40),Y(40),U(40),P(40),E(40),C(4,40),A(40,3),B(40),Z(40)
1)
  MM=M-1
20 FORMAT(1H1,T9*SUBROUTINE SPLICON DIFFERENCES,*/T11,D(K)=X(K+1)-X(K)
1)*T11,E(K)=(Y(K+1)-Y(K))/D(K)*T10,D(K)*T25,E(K)*
  DO 2 K=1,MM
    D(K)=X(K+1)-X(K)
    RLK CHECK. GUESSING AT 1.0E-8 AS A LIMIT.
    IF(D(K).LT.1.0E-8) STOP5
    P(K)=D(K)/6.
    E(K)=(Y(K+1)-Y(K))/D(K)
21 FORMAT(2F15.8)
2 CONTINUE
  DO 3 K=2,MM
    B(K)=E(K)-E(K-1)
    A(1,2)=-1.-D(1)/D(2)
    A(1,3)=D(1)/D(2)
    A(2,3)=P(2)-P(1)*A(1,3)
    A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
    A(2,3)=A(2,3)/A(2,2)
    R(2)=B(2)/A(2,2)
  DO 4 K=3,MM
    A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
    B(K)=B(K)-P(K-1)*B(K-1)
    A(K,3)=P(K)/A(K,2)
    B(K)=B(K)/A(K,2)
    Q=D(M-2)/D(M-1)
    A(M,1)=1.+Q+A(M-2,3)
    A(M,2)=-Q-A(M,1)*A(M-1,3)
    B(M)=B(M-2)-A(M,1)*B(M-1)
    Z(M)=B(M)/A(M,2)
  MN=M-2
  DO 6 I=1,MN
    K=M-I
    Z(K)=B(K)-A(K,3)*Z(K+1)
    Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
50 FORMAT( /,T5*SUBROUTINE SPLICON*/
1T0*X INPUT*,T22*Y INPUT*,T32*Z=2ND DERIV*/(1P3E14.5))
  DO 7 K=1,MM
    O=1./(6.*D(K))
    C(1,K)=Z(K)*O
    C(2,K)=Z(K+1)*Q
    C(3,K)=Y(K)/D(K)-Z(K)*P(K)
    C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
  RETURN
END

```

```

C
C
C
C
C
C
C
SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT,DERIV,IOK)
  INPUT X AND Y TABLE ARRAYS IN ASCENDING VALUES OF X
  INPUT M = NUMBER OF (X,Y) PAIRS IN TABLE
  INPUT C ARRAY OF CONSTANTS FROM SPLCON SUBROUTINE
  INPUT XINT TO INTERPOLATE
  OUTPUT YINT = INTERPOLATED VALUE
  OUTPUT IOK, IOK=1 IF (XINT-GE,X(1),AND,XINT,LT,X(M))
  IOK=0 IF XINT OUTSIDE RANGE. YINT AND DERIV SET EQUAL TO 0.0
  DIMENSION X(40),Y(40),C(4,40)
  IF(XINT-X(1))17,1,2
1 YINT=Y(1)
  GO TO 8
2 K=1
3 IF(XINT-X(K+1))6,4,5
4 YINT=Y(K+1)
  GO TO 8
5 K=K+1
  IF(M-K)7,7,3
6 YINT=(X(K+1)-XINT)*(C(1,K)*X(K+1)-XINT)**2+C(3,K))
  YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
  COMPUTATION OF DSTART FOR TIEVERIN = VALUE OF FIRST DERIVATIVE
  DERIVATIVE IS CALCULATED OVER SAME RANGE AS YINT
8 DERIV=-3.*C(1,K)*(X(K+1)-XINT)**2+3.*C(2,K)*(XINT-X(K))**2-C(3,K)+
  1C(4,K)
  IOK=1
  RETURN
7 IOK=0
  YINT=0.
  DERIV=0.
  RETURN
  END
C
C

```

```

PROGRAM CABEPSL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE16,
1TAPE17)
DIMENSION A(17),JFILE(45),XVNPRI(200),XGAMA(200),XRPR(200),
1XSPLNI(400),YSPLI(400),YSPL2(400),XTEMP(400),YTI(400),YT2(400),
1C1(4,200),C2(4,200),GAMA(400),RPR(400),SIGMA(400),EPS(400),
1DEPSDG(400),DSIGDG(400),DZPRDZ(400),VN(400),PHI(400),VNPRI(400)
** NOTE USE OF FBSFL ROUTINE *****
CALL FBSFL(17,32,0)
NTRU=0
READ(15,110) NFILES
READ(15,110) (JFILE(N),N=1,NFILES)
110 FORMAT(24I3)
PI=3.14159265358979
TWOP1=2.*PI
NF=0
111 N=0
112 NF=NF+1
113 N=N+1
120 READ(16,120) (A(I),I=1,7)
120 FORMAT(7E15.5)
CALL ENDFIL(16,1STAT)
1STAT = 1 FOR EOF, 2 FOR NO EOF.
IF(1STAT.EQ.1) GO TO 130
125 READ(16,120) (A(I),I=8,14),XVNPRI(N),XGAMA(N),XRPR(N)
X=A(2)
GO TO 112
130 DO 132 I=1,NFILES
130 IF(NF.EQ.JFILE(I)) GO TO 135
132 CONTINUE
GO TO 111
135 NPIS=N-1
DO 140 N=1,NPTS
135 IF(XGAMA(N).GT.1.0) GO TO 140
GO TO 142
140 CONTINUE
STOP
142 MIN=N
MAX=N-1
11=0
NBAD=0
DO 145 N=MIN,NPTS
11=11+1
XTEMP(11)=XGAMA(N)
Y1(11)=XRPR(N)
Y2(11)=XVNPRI(N)
IF(11.EQ.1) GO TO 145
IF(XTEMP(11).GT.XTEMP(11-1)) GO TO 145
NBAD=NBAD+1
11=11-1
WRITE(6,143) N,XGAMA(11),XRPR(11),XVNPRI(11)
143 FORMAT(//) BAD VALUES FOR SPLINE FIT. GAMA NOT INCREASING AT N=
143/* GAMA=1PE15.5,3X,*RPR=1PE15.5,3X,*VNPRI=1PE15.5)

```

```

145 CONTINUE
DO 150 N=1,MAX
  I1=I1+1
  XTEMP(I1)=XGAMA(N)
  Y1(I1)=XRPR(N)
  Y2(I1)=XVNP(N)
  IF(XTEMP(I1).GT.XTEMP(I1-1)) GO TO 150
  XSAO=XH/D+1
  I1=I1-1
  IF(N.EQ.1) GO TO 150
  WRITE(6,143) N,XGAMA(N),XRPR(N),XVNP(N)
150 CONTINUE
  M=APTS-N*QAD
  DO 150 I=1,3
    NH=I-(3-I)
    XSPLIN(I)=-(TWOPI-XTEMP(NN))
    YSPL1(I)=Y1(NN)
    YSPL2(I)=Y2(NN)
    XSPLIN(M+3+I)=TWOPI+XTEMP(I)
    YSPL1(M+3+I)=Y1(I)
    YSPL2(M+3+I)=Y2(I)
160 CONTINUE
  DO 165 I=1,M
    XSPLIN(I+3)=XTEMP(I)
    YSPL1(I+3)=Y1(I)
    YSPL2(I+3)=Y2(I)
  165 MPAIR=M+6
  *** NOTE PRINT-OUT *****
  WRITE(6,170) X,XSPLIN(1),YSPL1(1),YSPL2(1),I1=1,MPAIR)
170 FORMATT(1H,*, ORDERED VALUES GOING INTO SPLINE FIT. X=F6,3//12*G
    1AMA=Y28*RRPR*,T42*VNPR*(1P3E15.5))
    CALL SPLICON(XSPLIN,YSPL1,MPAIR,C1)
    CALL SPLICON(XSPLIN,YSPL2,MPAIR,C2)
    NBIG=180
    N1=NBIG+1
    DELTGA=2.*PI/(FLOAT(NBIG))
    D2=2.*DELTGA
    DELT2=DELTGA/2.
    DO 105 N=1,N1
      GAMA(N)=DELTGA*FLOAT(N-1)
      XINT=GAMA(N)
      CALL SPLINE(XSPLIN,YSPL1,MPAIR,C1,XINT,YINT,DUM,IOK)
      IF(IOK.EQ.0) WRITE(6,115) XINT
115 FORMAT(//*, OUT OF RANGE FOR SPLINE FIT/* GAMA=1PE15.5)
      IF(IOK.EQ.0) STOP2
      RPR(N)=YINT
      CALL SPLINE(XSPLIN,YSPL2,MPAIR,C2,XINT,YINT,DUM,IOK)
      IF(IOK.EQ.0) STOP2
      VNPR(N)=YINT
105 CONTINUE
  T=.1
  R=T*SQRT(1.-X**2)

```



```

DO 108 N=1,N1
108 SIGMA(N)=ALOG(RPR(N)/R)
DSIGDG(1)=(SIGMA(2)-SIGMA(NB1G))/D2
DO 10 N=2,NB1G
DSIGDG(N)=(SIGMA(N+1)-SIGMA(N-1))/D2
10 CONTINUE
DO 25 N=2,NB1G
EN=N
F1=4.*DEL TGA*DSIGDG(N)
SUM1=0.
NM1=N-2
IF(N.EQ.2) GO TO 20
DO 15 K=1,NM1
AK=K
SIGK=SIGMA(K+1)+SIGMA(K)
S1=SIN(DEL T2*(AK-EN+1.))
S2=SIN(DEL T2*(AK-EN))
SUM1=SUM1+SIGK*ALOG(S1/S2)
15 CONTINUE
20 SUM2=0.
IF (N.GE.NB1G) GO TO 23
NK=N+1
DO 24 K=NK,NB1G
AK=K
SIGK=SIGMA(K+1)+SIGMA(K)
S1=SIN(DEL T2*(AK-EN+1.))
S2=SIN(DEL T2*(AK-EN))
SUM2=SUM2+SIGK*ALOG(S1/S2)
24 CONTINUE
23 EPS(N)=-(P1+SUM1+SUM2)/(2.*P1)
25 CONTINUE
SUM3=0.
NM2=NB1G-1
DO 30 K=2,NM2
SIGK=SIGMA(K+1)+SIGMA(K)
S1=SIN(DEL T2*FLOAT(K))
S2=SIN(DEL T2*FLOAT(K-1))
SUM3=SUM3+SIGK*ALOG(S1/S2)
30 CONTINUE
EPS(1)=-(4.*CEL TGA*DSIGDG(1)+SUM3)/(2.*P1)
EPS(N1)=EPS(1)
DSIGDG(N1)=DSIGDG(1)
DO 50 K=1,N1
PHI(K)=GAMA(K)+EPS(K)
50 CONTINUE
SUM 4=0
DO 60 K=1,NM2
SUM4=SUM4+SIGMA(K+1)
60 CONTINUE
SIGMAZ=(SIGMA(1)+SUM4)*DEL TGA/(2.*P1)
DO 70 K=2,N

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```

DEPSDG(K)=(EPS(K+1)-EPS(K-1))/D2
70  CONTINUE
DEPSDG(1)=(EPS(2)-EPS(NBIG))/D2
DEPSDG(N1)=DEPSDG(1)
DO 80 N=1,N1
DEPRDZ(N)=(EXP(SIGMA(N)-SIGMAZ))*(SQRT(1+DSIGDG(N)**2))/
1(1+DEPSDG(N))
VN(N)=VNPR(N)*DZPRDZ(N)
80  CONTINUE
WRITE(6,200) X,SIGMAZ
200  FORMAT(1H1* X=F7.3,X,SIGMAZ=,1PE14.2//T4*N*,T11*GA1A*,T26*
1*RPR*,T38*SIGMA*,T54*EPS*,T65*DEPSDG*,T79*DSIGDG*,T93*DZPRDZ*,
1T11*VN*,11X*PHI*)
WRITE(6,201)(N,GAMA(N),RPR(N),SIGMA(N),EPS(N),DEPSDG(N),DSIGDG(N),
1DZPRDZ(N),VN(N),PHI(N),N=1,N1)
201  FORMAT(14,1P9E14.5)
WRITE(17,202) X,SIGMAZ
202  FORMAT(1P2E14.5)
WRITE(17,203)(N,GAMA(N),RPR(N),SIGMA(N),EPS(N),DEPSDG(N),
1DSIGDG(N),DZPRDZ(N),VN(N),PHI(N),N=1,N1)
ENDFILE 1,
NTHRU=NTHRU+1
IF(NTHRU.LT.NFILES) GO TO 111
STOP
END

```

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PROGRAM CASVINT(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE17)
DIMENSION XGAMA(800),XRPR(800),XVN(800),XPHI(800),C1(4,200),
IC2(4,200),C3(4,200),XSPLIN(200),YSPL1(200),YSPL2(200),YSPL3(200),
IGAMA(800),RPR(800),PHI(800),VS(800),DVNDPH(800),VN(800),Z1(800),
I22(800),FY(50),FZ(50),XX(50)
PI=3.14159265358979
TWOPI=2.*PI

***
READ(5,500) (XX(ICNT),FZ(ICNT),FY(ICNT),ICNT=1,35)
500 FORMAT(F6.3,2E11.5)
ICNT=35
201 PEAD(17,202) X,SIGMAZ
202 FORMAT(2E14.5)
N=0
125 N=N+1
READ(17,120) M,XGAMA(N),XRPR(N),DUM,DUM,DUM,DUM,XV:(N),XPHI(N)
120 FORMAT(14,9E14.5)
IF(LOF(17)) 130,125
130 NPTS=N-1
***
IF(X.LT.-.120) GO TO 201
DO 160 I=1,2
NN=NPTS-(3-I)
XSPLIN(1)=XPHI(NN)-TWOPI
YSPL1(1)=XVN(NN)
YSPL2(1)=XRPR(NN)
YSPL3(1)=XGAMA(NN)-TWOPI
XSPLIN(NPTS+2)=XPHI(I+1)+TWOPI
YSPL1(NPTS+2)=XVN(I+1)
YSPL2(NPTS+2)=XRPR(I+1)
YSPL3(NPTS+2)=XGAMA(I+1)+TWOPI
160 CONTINUE
DO 165 I=1,NPTS
XSPLIN(I+2)=XPHI(I)
YSPL1(I+2)=XVN(I)
YSPL2(I+2)=XRPR(I)
165 YSPL3(I+2)=XGAMA(I)
MPAIR=NPTS+4
WRITE(5,163) X
168 FORMAT(1H1* ORDERED VALUES GOING INTO SPLINE FIT AT X=F6.3/
ITIN*PI*,I34*VN*,T48*RPR*,T62*XGAMA*)
DO 170 N=1,MPAIR
WRITE(6,169) XSPLIN(N),YSPL1(N),YSPL2(N),YSPL3(N)
169 FORMAT(5X,1P4E15.5)
IF(N.EQ.1) GO TO 170
IF(XSPLIN(N).LE.XSPLIN(N-1)) STOP1
170 CONTINUE
CALL SPLCON(XSPLIN,YSPL1,MPAIR,C1)
CALL SPLCON(XSPLIN,YSPL2,MPAIR,C2)
CALL SPLCON(XSPLIN,YSPL3,MPAIR,C3)
MBIG=180
JJ=0
175 MI=NRIG+1

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DLTPHI=2.*PI/FLOAT(MBIG)
D3=2.*DLTPHI
DELT3=DLTPHI/2.
DO5M=1,M1
PHI(M)=DLTPHI*FLOAT(M-1)
5  CONTINUE
DO 6 M=1,M1
XINT=PHI(M)
CALL SPLINE(XSPLIN,YSPL1,MPAIR,C1,XINT,YINT,DERIV,IOK)
IF(IOK.EQ.0) WRITE(6,115) XINT
115 FORMAT(//'* OUT OF RANGE FOR SPLINE FIT. PHI=1PE14.5)
IF(IOK.EQ.0) STOP2
VN(M)=YINT
CALL SPLINE(XSPLIN,YSPL2,MPAIR,C2,XINT,YINT,DERIV,IOK)
IF(IOK.EQ.0) STOP2
RPR(M)=YINT
CALL SPLINE(XSPLIN,YSPL3,MPAIR,C3,XINT,YINT,DERIV,IOK)
IF(IOK.EQ.0) STOP2
GAMA(M)=YINT
6  CONTINUE
LVNDPH(1)=(VN(2)-VN(MBIG))/D3
DO10 M=2,MBIG
DVNDPH(M)=(VN(M+1)-VN(M-1))/D3
10  CONTINUE
DO 25 M=2,MBIG
EM=M
P1=4.*DLTPHI*DVNDPH(M)
SUM1=0
MA1=M-2
IF(M.EQ.2) GO TO 20
DO 15 K=1,MA1
AK=K
VNK=VN(K+1)+VN(K)
S1=SIN(DELT3*(AK-EM+1.))
S2=SIN(DELT3*(AK-EM))
SUM1=SUM1+VNK*ALOG(S1/S2)
15  CONTINUE
20  SUM2=0
IF(M.GE.MBIG) GO TO 23
NK=M+1
DO 24 K=MK,MBIG
AK=K
VNK=VN(K+1)+VN(K)
S1=SIN(DELT3*(AK-EM+1.))
S2=SIN(DELT3*(AK-EM))
SUM2=SUM2+VNK*ALOG(S1/S2)
24  CONTINUE
23  VS(M)=-(P1+SUM1+SUM2)/(2.*PI)
25  CONTINUE
SUM3=0
MA2=MRIG-1
DO30K=2,MA2
VNK=VN(K+1)+VN(K)
S1=SIN(DELT3*FLOAT(K))

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S2=SIN(DELTA3*FLOAT(K-1))
SUM3=SUM3+V/NK*ALOG(S1/S2)
30  CONTINUE
VS(1)=-[4.*DLTPH1*DVNDPH(1)+SUM3]/(2.*P[1])
VS(M1)=VS(1)
DVNDPH(M1)=DVNDPH(1)
WRITE(6,310) X
310  FORMAT(1H1,T5*X=F6.3/T5*N*,T14*PHI*,T30*VN*,T45*VS*,T59*RRPR*,T73*
1*GAMA*,T716*HRPR*VS*COS(GAMA),T4X16*HRPR*VS*SIN(GAMA))
DO 330 N=1,M1
Z1(N)=RPR(N)*VS(N)*COS(GAMA(N))
Z2(N)=RPR(N)*VS(N)*SIN(GAMA(N))
WRITE(6,320) N,PHI(N),VN(N),VS(N),RPR(N),GAMA(N),Z1(N),Z2(N)
320  FORMAT(15,1P5E15.5,E17.5,E20.5)
330  CONTINUE
SUM1=(Z1(1)+Z1(M1))/2.
SUM2=(Z2(1)+Z2(M1))/2.
DO 340 N=2,M1G
SUM1=SUM1+Z1(N)
SUM2=SUM2+Z2(N)
Z1INT=SUM1*DLTPH1
WRITE(6,345) Z1INT
345  FORMAT(//79H  INTEGRAL OF (RPR*VS*COS(GAMA)DPHI) =,1PE13.5)
Z2INT=SUM2*DLTPH1
WRITE(6,346) Z2INT
346  FORMAT(39H  INTEGRAL OF (RPR*VS*SIN(GAMA)DPHI) =,1PE13.5)
T=.1
R=T*SQRT(1.0-X**2)
CONST=2.*R*EXP(SIGMA2)
WRITE(6,350) CONST
350  FORMAT(1*  CONSTANT =,1PE13.5)
ICNT=ICNT+1
FZ(ICNT)=CONST*Z1INT
FY(ICNT)=-CONST*Z2INT
XX(ICNT)=X
WRITE(6,355) FZ(ICNT),FY(ICNT)
355  FORMAT(1*  FZ =,1PE13.5/*  FY =,1PE13.5)
IF(X.GT.0.45) GO TO 420
IF(ARS(X-0.1).GT.1.0F-8) GO TO 201
X=-.970*-.950*-.920*-.900*-.875*-.825.ETC.
FTRAP=.020*(FY(1)+FY(2))/2.
FTRAP=FTRAP+.030*(FY(2)+FY(3))/2.
FTRAP=FTRAP+.030*(FY(3)+FY(4))/2.
DO 410 I=5,ICNT
410  FTRAP=FTRAP+.025*(FY(I-1)+FY(I))/2.
XMY=-FY(ICNT)+FTRAP
WRITE(6,415) FTRAP,XMY
415  FORMAT(1*  FY INTEGRAL =,1PE13.5/*  MOMENT ABOUT NOSE =,1PE13.5)
GO TO 201
420  WRITE(6,421) (XX(I),FZ(I),FY(I),I=1,ICNT)
421  FORMAT(1H1,T11*X*,T25*FZ*,T40*FY*/(1P3E15.5))
WRITE(6,425) XMY

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425 FORMAT(/* MOMENT ABOUT NOSE =#1PE13.5)
400 STOP
END